

# **Bin Packing and Cutting Stock Problems: Mathematical Models and Exact Algorithms**

Maxence Delorme<sup>(1)</sup>, Manuel Iori<sup>(2)</sup>, Silvano Martello<sup>(1)</sup>

<sup>(1)</sup>*DEI “Guglielmo Marconi”, University of Bologna*

<sup>(2)</sup>*DISMI, University of Modena and Reggio Emilia*

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DEI ”GUGLIELMO MARCONI”, UNIVERSITY OF BOLOGNA

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Maxence Delorme<sup>(1)</sup>, Manuel Iori<sup>(2)</sup>, Silvano Martello<sup>(1)</sup>

(1) *DEI "Guglielmo Marconi", University of Bologna*

(2) *DISMI, University of Modena and Reggio Emilia*

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## Abstract

We review the most important mathematical models and algorithms developed for the exact solution of the one dimensional bin packing and cutting stock problems, and experimentally evaluate, on state-of-the art computers, the performance of the main available software tools.

**Keywords:** Bin packing, Cutting stock, Exact algorithms, Computational evaluation.

## 1 Introduction

The (one-dimensional) bin packing problem is one of the most famous problems in combinatorial optimization. Its structure and its applications have been studied since the thirties, see Kantorovich [80]. In 1961 Gilmore and Gomory [67] introduced, for this class of problems, the concept of column generation, by deriving it from earlier ideas of Ford and Fulkerson [61], and Dantzig and Wolfe [39]. This is one of the first problems for which, since the early seventies, the worst-case performance of approximation algorithms was investigated. In the next decades lower bounds were studied, and exact algorithms proposed. As the problem is strongly  $\mathcal{NP}$ -hard, many heuristic and metaheuristic approaches have also been proposed along the years. The purpose of this article is to review the most important mathematical models and algorithms developed for the exact solution of the problem, and to experimentally evaluate, on state-of-the art computers, the performance of the main available software tools.

The *bin packing problem* (BPP) can be informally defined in a very simple way. We are given  $n$  items, each having an integer *weight*  $w_j$  ( $j = 1, \dots, n$ ), and an unlimited number of identical *bins* of integer *capacity*  $c$ . The objective is to pack all the items into the minimum number of bins so that the total weight packed in any bin does not exceed the capacity. (In a different but equivalent *normalized* definition, the weights are real numbers in  $[0, 1]$ , and the capacity is 1.) We assume, with no loss of generality, that  $0 < w_j < c$  for all  $j$ .

Formally, let  $u$  be any upper bound on the optimal solution value (for example, the value of any approximate solution), and assume that the potential bins are numbered as  $1, \dots, u$ . By introducing two binary decision variables

$$y_i = \begin{cases} 1 & \text{if bin } i \text{ is used in the solution;} \\ 0 & \text{otherwise} \end{cases} \quad (i = 1, \dots, u), \quad (1)$$

$$x_{ij} = \begin{cases} 1 & \text{if item } j \text{ is packed into bin } i; \\ 0 & \text{otherwise,} \end{cases} \quad (i = 1, \dots, u; j = 1, \dots, n), \quad (2)$$

we can model the BPP as an *Integer Linear Program* (ILP) of the form (see Martello and Toth [105])

$$\min \sum_{i=1}^u y_i \quad (3)$$

$$\text{s.t.} \quad \sum_{j=1}^n w_j x_{ij} \leq cy_i \quad (i = 1, \dots, u), \quad (4)$$

$$\sum_{i=1}^u x_{ij} = 1 \quad (j = 1, \dots, n), \quad (5)$$

$$y_i \in \{0, 1\} \quad (i = 1, \dots, u), \quad (6)$$

$$x_{ij} \in \{0, 1\} \quad (i = 1, \dots, u; j = 1, \dots, n). \quad (7)$$

Constraints (4) impose that the capacity of any used bin is not exceeded, while constraints (5) ensure that all the items are packed.

Many variants and generalization of the BPP arise in practical contexts. One of the most important applications, studied since the sixties, is the *Cutting Stock Problem* (CSP). Although it has been defined in different ways according to specific real world cases, its basic definition, using the BPP terminology, is as follows. We are given  $m$  *item types*, each having an integer *weight*  $w_j$  and an integer *demand*  $d_j$  ( $j = 1, \dots, m$ ), and an unlimited number of identical *bins* of integer *capacity*  $c$ . (In the CSP literature the bins are frequently called *rolls*, the term coming from early applications in the paper industry, and “cutting” is normally used instead of “packing”.) The objective is to produce  $d_j$  copies of each item type  $j$  (i.e., to cut/pack them) using the minimum number of bins so that the total weight in any bin does not exceed the capacity. Formally, let us define  $u$  and  $y_i$  as above, and let

$$\xi_{ij} = \text{number of items of type } j \text{ packed into bin } i \quad (i = 1, \dots, u; j = 1, \dots, m). \quad (8)$$

The CSP is then

$$\min \sum_{i=1}^u y_i \quad (9)$$

$$\text{s.t.} \quad \sum_{j=1}^m w_j \xi_{ij} \leq cy_i \quad (i = 1, \dots, u), \quad (10)$$

$$\sum_{i=1}^u \xi_{ij} = d_j \quad (j = 1, \dots, m), \quad (11)$$

$$y_i \in \{0, 1\} \quad (i = 1, \dots, u), \quad (12)$$

$$\xi_{ij} \geq 0, \text{ integer} \quad (i = 1, \dots, u; j = 1, \dots, m). \quad (13)$$

The BPP can be seen as a special case of the CSP in which  $d_j = 1$  for all  $j$ . In turn, the CSP can be modeled by a BPP in which the item set includes  $d_j$  copies of each item type  $j$ .

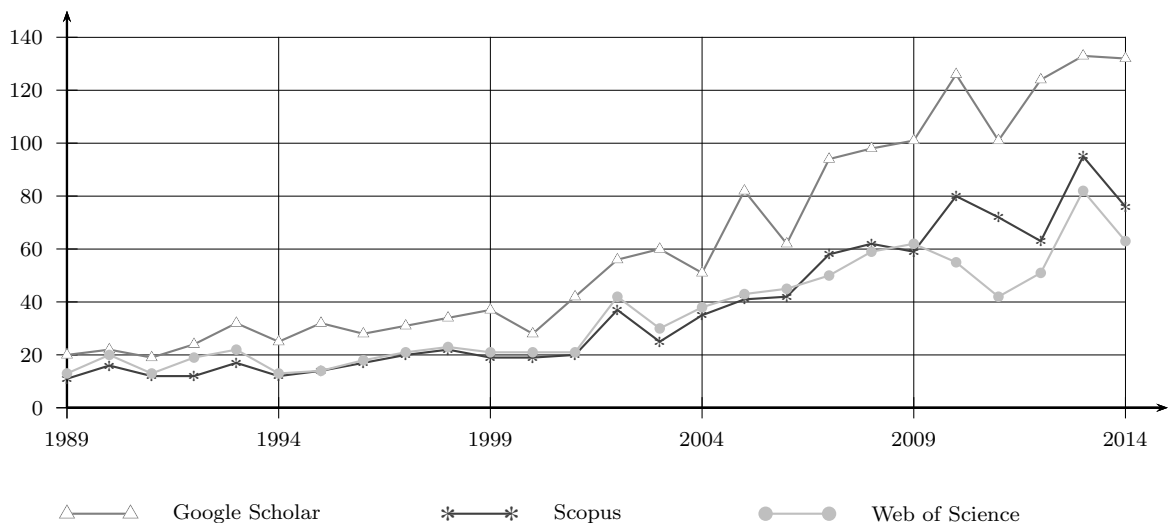
The BPP (and hence the CSP) has been proved to be  $\mathcal{NP}$ -hard in the strong sense by Garey and Johnson [63] through transformation from the 3-Partition problem. The literature on these problems and their variants (including extensions to two- and three-dimensional cases) is huge. Figure 1 shows the number of articles having in the title either the term "bin packing", or the term "cutting stock", or both, according to different bibliographic data bases, in the years 1989-2014. The picture shows the growing interest in these problems, with sharp increase in recent years.

This paper is devoted to a presentation of the main mathematical models that have been proposed, and to an experimental evaluation of the main available software tools that have been developed. For exhaustive studies on specific research areas concerning the BPP and the CSP, the reader is referred to many surveys that have been published along the years. To the best of our knowledge, the following reviews have been proposed.

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**Figure 1** Number of papers on the BPP and the CSP, 1989-2014.

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The first literature review on these problems was published in 1992 by Sweeney and Paternoster [138], who collected more than 400 books, articles, dissertations, and working papers appeared from 1961 to 1990. In 1990 Dyckhoff [48] had proposed a typology of cutting and packing problems, and classified the BPP and the CSP as 1/V/I/M and 1/V/I/R, respectively. In the same year Martello and Toth included a chapter on the BPP in their book [105] on knapsack problems. Two years later Dyckhoff and Finke [49] published a book on cutting and packing problems arising in production and distribution, where they investigated the different structure of these problems, and classified the literature accordingly. A bibliography on the BPP has been compiled by Coffman, Csirik, Johnson, and Woeginger [33]. More recently, Wäscher, Haußner, and Schumann [151] re-visited the typology by Dyckhoff [48] and proposed more detailed categorization criteria.

Besides the general surveys discussed above, a number of reviews concerning specific methodologies have been proposed. Already in the early eighties Coffman, Garey, and Johnson [64, 35] presented surveys on approximation algorithms for the BPP. Other surveys on approximation algorithms for the BPP and a number of its variants were later proposed by Coffman, Garey, and Johnson [36], Coffman, Galambos, Martello, and Vigo [34], and Coffman and Csirik [31]. Coffman and Csirik [30] also proposed a four-field classification scheme for papers on bin packing, aimed at highlighting the results in bin packing theory to be found in a certain article. More recently, Coffman, Csirik, Galambos, Martello, and Vigo [32] presented an overview of approximation algorithms for the BPP and a number of its variants, and classified all references according to [30].

Valério de Carvalho [143] presented a survey of the most popular *Linear Programming* (LP) methods for the BPP and the CSP. A review of models and solution methods was included by Belov [9] in his PhD thesis dedicated to one- and two-dimensional cutting stock problems.

We finally mention that extensions to higher dimensions have been investigated too. In the early nineties, Haessler and Sweeney [75] provided a description of one- and two-dimensional cutting stock problems, and a review of some of the methods to solve them. More recently, surveys on two-dimensional packing problems have been presented by Lodi, Martello, Monaci, and Vigo [94, 96, 95].

In the next section we briefly review the most successful upper and lower bounding techniques for the considered problems. In Sections 3, 4, and 5 we examine pseudo-polynomial formulations, branch-and-bound algorithms, and branch-and-price approaches, respectively. Finally, in Section 6, we experimentally evaluate the computational performance of twelve computer programs available for the solution of the considered problems. Conclusions follow in Section 7.

## 2 Upper and lower bounds

Most exact algorithms for bin packing problems make use of upper and lower bound computations in order to guide the search in the solution space, and to fathom partial solutions that cannot lead to optimal ones. As previously mentioned, for deep reviews on these specific domains, the reader is referred to the surveys listed in the previous section. In this section we briefly review the most successful upper and lower bounding techniques that have been

developed, with some focus on areas for which no specific survey is available. We use the term *approximation algorithm* for methods for which theoretical results (like, e.g., worst-case performance) can be established, while the term *heuristic* will denote methods for which the main interest relies in their practical behavior.

A classical way for evaluating upper and lower bounds is their absolute worst-case performance ratio. Given a minimization problem and an approximation algorithm  $A$ , let  $A(I)$  and  $OPT(I)$  be the solution value provided by  $A$  and the optimal solution value, respectively, for an instance  $I$  of the problem. The *worst-case performance ratio* (WCPR) of  $A$  is then defined as the smallest real number  $\bar{r}(A) > 1$  such that  $A(I)/OPT(I) \leq \bar{r}(A)$  for all instances  $I$ , i.e.,

$$\bar{r}(A) = \sup_I \{A(I)/OPT(I)\}. \quad (14)$$

Similarly, the WCPR of a lower bound  $L$  is the largest real number  $\underline{r}(L) < 1$  such that, for all instances  $I$ , the lower bound value  $L(I)$  satisfies  $L(I)/OPT(I) \geq \underline{r}(L)$ , i.e.,

$$\underline{r}(L) = \inf_I \{L(I)/OPT(I)\}. \quad (15)$$

## 2.1 Approximation algorithms

The simplest BPP approximation algorithms consider the items in any sequence. Algorithm *Next-Fit* (NF) at each iteration packs the next item into the current bin (initially, into bin 1) if it fits, or into a new bin (which becomes the current one) if it does not fit. The WCPR of NF is  $\bar{r}(NF) = 2$ . Algorithm *First-Fit* (FF) at each iteration packs the next item into the lowest indexed bin where it fits, or into a new bin if it does not fit in any open bin. Algorithm *Best-Fit* (BF) at each iteration packs the next item into the feasible bin (if any) where it fits by leaving the smallest residual space, or into a new one if no open bin can accommodate it. The exact WCPR of FF and BF has been an open problem for forty years, until recently Dósa and Sgall [44, 45] proved that  $\bar{r}(FF) = \bar{r}(BF) = \frac{17}{10}$ .

Better performances are obtained by preventively sorting the items according to non-increasing weight. The WCPR of the resulting algorithms *First-Fit Decreasing* (FFD) and *Best-Fit decreasing* (BFD) is  $\bar{r}(FFD) = \bar{r}(BFD) = \frac{3}{2}$  (Simchi-Levi [134]). Moreover, this is the best achievable performance, in the following sense:

**Property 1** *No polynomial-time approximation algorithm for the BPP can have a WCPR smaller than  $\frac{3}{2}$  unless  $\mathcal{P} = \mathcal{NP}$ .*

**Proof** Consider an instance of the  $\mathcal{NP}$ -complete PARTITION problem: is it possible to partition  $S = \{w_1, \dots, w_n\}$  into  $S_1, S_2$  so that  $\sum_{j \in S_1} w_j = \sum_{j \in S_2} w_j$ ? Assume a polynomial-time approximation algorithm  $A$  for the BPP exists such that  $OPT(I) > \frac{2}{3} A(I)$  for all instances  $I$ , and execute  $A$  for an instance  $\hat{I}$  of the BPP defined by  $(w_1, \dots, w_n)$  and  $c = \sum_{j=1}^n w_j/2$ . If  $A(\hat{I}) = 2$  then we know that the answer to PARTITION is **yes**. If instead  $A(\hat{I}) \geq 3$  then we know that  $OPT(\hat{I}) > \frac{2}{3} 3$ , i.e., that  $OPT(\hat{I}) > 2$ , hence the answer to PARTITION is **no**. It follows that we could solve PARTITION in polynomial time.  $\square$

Since FFD and BFD provide the best possible WCPR, most research on approximation algorithms for the BPP focused on the *asymptotic* WCPR, defined as the minimum real

number  $\bar{r}^\infty(A)$  such that, for some positive integer  $k$ ,  $A(I)/OPT(I) \leq \bar{r}^\infty(A)$  for all instances  $I$  satisfying  $OPT(I) \geq k$ . The number of results in this area is impressive, and beyond the purpose of this study: we refer the reader to the various surveys that were listed in Section 1. The most recent survey (2013), by Coffman, Csirik, Galambos, Martello, and Vigo [32], examines 200 references from the literature. Among the papers that appeared subsequently, we mention those by Dósa, Li, Han, and Tuza [43] on the FFD algorithm, by Rothvoß [120], who improved a classical result by Karmarkar and Karp [82], and by Balogh, Békési, Dósa, Sgall, and van Stee [6], who closed a long standing open issue on on-line bin packing.

## 2.2 Lower bounds

To our knowledge, no general survey on lower bounds for the BPP is available. Hence we provide in the following a brief review of the corresponding literature. An obvious lower bound for the BPP, computable in  $O(n)$  time, is provided by the so-called *continuous relaxation*, namely

$$L_1 = \left\lceil \sum_{j=1}^n w_j/c \right\rceil, \quad (16)$$

which gives the rounded solution value of the linear programming relaxation of (3)-(7). It is easily seen that  $\underline{L}(L_1) = \frac{1}{2}$  (see, e.g., Martello and Toth [105]).

A better lower bound was obtained by Martello and Toth [104]. Given any integer  $\alpha$  ( $0 \leq \alpha \leq c/2$ ), let

$$\begin{aligned} J_1 &= \{j \in N : w_j > c - \alpha\}; \\ J_2 &= \{j \in N : c - \alpha \geq w_j > c/2\}; \\ J_3 &= \{j \in N : c/2 \geq w_j \geq \alpha\}, \end{aligned}$$

and observe that each item in  $J_1 \cup J_2$  needs a separate bin, and that no item of  $J_3$  can go to a bin containing an item of  $J_1$ . Then  $L(\alpha) = |J_1| + |J_2| + \max\left(0, \left\lceil \frac{\sum_{j \in J_3} w_j - (|J_2|c - \sum_{j \in J_2} w_j)}{c} \right\rceil\right)$  is a valid lower bound. It can be shown that the overall bound

$$L_2 = \max\{L(\alpha) : 0 \leq \alpha \leq c/2, \alpha \text{ integer}\} \quad (17)$$

can be computed in  $O(n \log n)$  time and has WCPR equal to  $\frac{2}{3}$ . Similarly to what happens for algorithms FFD and BFD, this is the best achievable performance, namely:

**Property 2** *No lower bound, computable in polynomial time, for the BPP can have a WCPR greater than  $\frac{2}{3}$  unless  $\mathcal{P} = \mathcal{NP}$ .*

**Proof** We will use the same instance of PARTITION as in the proof of Property 1, and the same induced BPP instance  $\hat{I}$ . Assume a polynomial-time lower bound  $L$  for the BPP exists such that  $OPT(I) < \frac{3}{2} L(I)$  for all instances  $I$ , and compute  $L$  for instance  $\hat{I}$ . If  $L(\hat{I}) \geq 3$  then we know that the answer to PARTITION is **no**. If  $L(\hat{I}) = 2$  then we know that  $OPT(\hat{I}) < \frac{3}{2} \cdot 2$ , hence  $OPT(\hat{I}) = 2$ , i.e., that the answer to PARTITION is **yes**. We could then solve PARTITION in polynomial time.  $\square$

Lower bounds that generalize  $L_2$  and can have better practical performance have been proposed by Labbé, Laporte, and Mercure [89] (lower bound  $L_{2LLM}$ ), and by Chen and Srivastava [26]. Theoretical properties of such bounds were studied by Elhedhli [53]. Bourjolly

and Rebetez [17] proved that the asymptotic WCPR of the bound  $L_{2LLM}$  proposed in [89] is  $\underline{r}^\infty(L_{2LLM}) = \frac{3}{4}$ .

Another lower bound,  $L_3$ , dominating  $L_2$  was obtained by Martello and Toth [104] by iteratively reducing the instance, and invoking  $L_2$  on the reduced instance. The time complexity grows to  $O(n^3)$ , and the asymptotic WCPR is  $\underline{r}^\infty(L_3) = \frac{3}{4}$ , as proved by Crainic, Perboli, Pezzuto, and Tadei [37].

A different type of lower bound computation had been considered in the eighties by Lueker [100], who proposed a bounding strategy for the case where all the items are drawn from a uniform distribution, based on dual feasible functions, which were originally introduced by Johnson [78]. Consider the normalized definition of the BPP (see Section 1): a real-valued function  $u(x)$  is called *dual feasible* if, for any finite set  $S$  of nonnegative real numbers, we have  $\sum_{x \in S} x \leq 1 \Rightarrow \sum_{x \in S} u(x) \leq 1$ . It follows that any lower bound computed over weights  $u(w)$  is also valid for the original weights  $w$ .

Later on, Fekete and Schepers [58] used dual feasible functions to produce new classes of fast BPP lower bounds. For example, given any normalized instance  $I$  of the BPP, any  $\alpha$  ( $0 \leq \alpha \leq 1/2$ ), and an item weight  $w$ , let  $w' = w/c$  and define

$$U^{(\alpha)}(w') = \begin{cases} 1 & \text{if } w' > 1 - \alpha; \\ w' & \text{if } 1 - \alpha \geq w' \geq \alpha; \\ 0, & \text{if } w' < \alpha. \end{cases}$$

Then  $U^{(\alpha)}(w')$  is a dual feasible function. (Observe in particular that, by considering all  $\alpha$  values in  $[0, \frac{1}{2}]$  and computing the corresponding bounds  $L_1$ , the maximum resulting value coincides with the value provided by  $L_2$ .)

A number of other dual feasible functions have been proposed in the literature. We refer the reader to Clautiaux, Alves, and Valério de Carvalho [28] for a recent survey on these functions and their use for the computation of BPP lower bounds, and to the recent article by Rietz, Alves, and Valério de Carvalho [113].

Chao, Harper, and Quong [25] and Crainic, Perboli, Pezzuto, and Tadei [38] studied methods for computing “fast” lower bounds for the BPP, i.e., bounds requiring no more than  $O(n \log n)$  time. Once a lower bound value, say  $\ell$ , has been computed, it can sometimes be improved through additional considerations: for example, if it can be established that no feasible solution using  $\ell$  bins exists, then  $\ell + 1$  is a valid lower bound value. Improvement techniques of this kind have been studied by Dell’Amico and Martello [42], Alvim, Ribeiro, Glover, and Aloise [2], Haouari and Gharbi [74], and Jarboui, Ibrahim, and Rebai [77].

Other very effective lower bounds which however require a non-polynomial time, including the famous column generation method by Gilmore and Gomory [67], will be discussed in Section 5.

## 2.3 Heuristics and metaheuristics

The focus of this work is on the optimal solution of bin packing and cutting stock problems. Approximate and heuristic solutions have thus marginal interest here, but they are commonly used to provide an initial solution to exact algorithms. For the sake of completeness, in this section we briefly review a number of heuristic and metaheuristic approaches.



## Heuristics

The first relevant contribution of this kind is probably the one by Eilon and Christofides [50] who presented a heuristic for a number of packing problems, basically consisting of algorithm BFD (see Section 2.1), plus a reshuffle routine when the solution is not equal to the continuous relaxation  $L_1$ . Roodman [117] presented a set of heuristics for variants of the CSP, mainly based on an initial greedy solution improved through local search. Vahrenkamp [141] proposed a random search for the CSP, based on a heuristic developed by Haessler [73] for generating cutting patterns for trim problems. Wäscher and Gau [150] considered a generalization of the CSP, and studied the computational behavior of heuristics based on rounding the solutions obtained from the LP relaxation of a generalization of the Gilmore and Gomory [67] model (see Section 5). The experiments were performed on random instances produced by their generator, CUTGEN (see Gau and Wäscher [65]), which creates CSP instances depending on five parameters: number of items, minimum and maximum weight, bin capacity, and average demand.

Gupta and Ho [72] proposed a heuristic algorithm based on the minimization of the unused bin capacities, and successfully compared it with FFD and BFD (although at the expenses of higher CPU times). Osogami and Okano [109] proposed variants of some classical approximation algorithms, and investigated the effect of a local search based on item exchanges. Other modifications of classical approximation algorithms were proposed by Bhatia, Hazra, and Basu [16], Kim and Wy [85], and Fleszar and Charalambous [59]. The effectiveness of a hill climbing local search strategy for the BPP, also based on item exchanges, was later investigated by Lewis [91].

As for most  $\mathcal{NP}$ -hard problems, starting from the early nineties many metaheuristic approaches of all kinds have been proposed for the BPP and the CSP. In the following, we list, grouped by metaheuristic paradigm, a number of contributions that provided interesting insights into the problems at hand.

## Simulated annealing and Tabu search

A classical simulated annealing approach to the BPP was implemented by Kämpke [79], while a variant of the method (called weight annealing) was proposed by Loh, Golden, and Wasil [97]. Scholl, Klein, and Jürgens [128] used a Tabu search procedure to speed up their well-known exact algorithm (BISON) for the BPP, treated in Section 4. Alvim, Ribeiro, Glover, and Aloise [2] embedded a Tabu search in a hybrid improvement heuristic for the BPP.

## Population based algorithms

Probably, the first genetic approach to the BPP is the one by Falkenauer and Delchambre [56]: they showed that the classical genetic approach cannot work efficiently for certain kinds of problems (like the BPP), and presented a variant (the grouping genetic algorithm) capable of producing a good computational behavior. Falkenauer [55] improved this method through hybridization with the dominance criterion by Martello and Toth [104] (see Section 4), and proposed a set of benchmark instances that was later adopted by many authors for computationally testing BPP algorithms. Although Gent [66] showed that the majority of them are very easy, these instances were used, e.g., for testing the genetic approaches by Reeves [112], Bhatia and Basu [15], Singh and Gupta [135], Ülker, Korkmaz, and Özcan

[140], and Stawowy [137]. Other genetic algorithms were proposed by Poli, Woodward, and Burke [110], and by Rohlfshagen and Bullinaria [115, 116]. Recently, a very effective genetic algorithm was proposed by Quiroz-Castellanos, Cruz-Reyes, Torres-Jimenez, Gómez, Fraire Huacuja, and Alvim [111].

Levine and Ducatelle [90] used an ant colony approach combined with a local search to solve the BPP. Liang, Yao, Newton, and Hoffman [92] proposed an evolutionary programming algorithm for the CSP and some of its variants.

### Hyper-heuristics

Ross, Schulenburg, Marín-Blázquez, and Hart [118, 119] attacked the BPP through combinations of genetic algorithms and hyper-heuristics. Other combinations of evolutionary algorithms and hyper-heuristics for the BPP were proposed by López-Camacho, Terashima-Marí, and Ross [98], Sim, Hart, and Paechter [133], and Burke, Hyde, and Kendall [20].

Bai, Blazewicz, Burke, Kendall, and McCollum [4] tested on BPP instances their simulated annealing hyper-heuristic approach. Sim and Hart [132] used genetic programming as a generative hyper-heuristic to create deterministic heuristics.

### Other meta-heuristic approaches

Fleszar and Hindi [60] obtained new heuristics for the BPP by modifying the heuristic of Gupta and Ho [72], and proposed a variable neighborhood search algorithm. Gómez-Meneses and Randall [70] proposed a hybrid extremal optimization approach with local search for the BPP.

## 3 Pseudo-Polynomial Formulations

The textbook BPP model (3)–(7) has its roots in the seminal work by Kantorovich [80]. Several attempts have been made since then to try and improve the computational behavior of the model, especially by providing families of valid inequalities. The simple inequality  $y_i \geq y_{i+1}$  for  $i = 1, \dots, u - 1$  reduces the size of the enumeration tree by imposing that the bins are used in increasing order of index. Symmetric solutions can be further removed by setting  $x_{ij} = 0$  for all  $j = 1, \dots, u - 1$  and  $i = j + 1, \dots, u$ , as there is always an optimal solution in which item 1 is packed in bin 1, item 2 either in bin 1 or 2, and so on. The linear relaxation of the model can be further strengthened by imposing that full items cannot be packed into bins  $i$  with fractional  $y_i$  value, i.e.,  $x_{ij} \leq y_i$  for all  $i = 1, \dots, u$  and  $j = 1, \dots, n$ . A number of enhanced families of inequalities, including the well-known *cover inequalities* and their generalizations, derive from studies on the knapsack polytope. For a detailed description of these inequalities, as well as of efficient separation procedures, we refer the reader to Gabrel and Minoux [62] and Kaparis and Letchford [81].

Despite these results, the computational behavior of model (3)–(7) remains quite poor, as it will be shown in Section 6. The literature has consequently focused on the study of models with better computational performance, including *pseudo-polynomial* models. The drawback of these models is that the number of variables depends not only on the number of items but also on the bin capacity. On the other hand, they provide a stronger linear relaxation than that given by (3)–(7).

In the next sections we present the main pseudo-polynomial models proposed in the literature, and highlight some relations among them. In Section 3.1 we address the oldest such model, introduced in 1981 by Dyckhoff [47]. The most relevant approach of this kind was presented in 1999 by Valério de Carvalho [142] for the CSP. In 2010, Cambazard and O’Sullivan [21] presented a BPP pseudo-polynomial model based on a similar idea, but described in a form inspired by the graph construction used by Trick [139] for propagating knapsack constraints. We anticipate its description in Section 3.2, since this makes it easier to understand the Valério de Carvalho model, which is then discussed in Section 3.3. The following example will be resumed a number of times in the next sections.

**Example 1** *For the BPP, we will consider an instance with  $n = 6$ ,  $c = 9$ , and  $w = (4, 4, 3, 3, 2, 2)$ . The equivalent CSP instance has  $m = 3$ ,  $c = 9$ ,  $w = (4, 3, 2)$ , and  $d = (2, 2, 2)$ . An optimal solution has value 2, and packs three items (of weight 4, 3, and 2) in each bin.  $\square$*

### 3.1 One-cut Formulation

The idea of the Dyckhoff [47] model for the CSP is to simulate the physical cutting process, by first dividing an ideal bin into two pieces (“left” and “right”), where the left piece is an item that has been cut, while the right piece is either a residual that can be re-used to produce other items or it is another item. The process is iterated on cutting residuals or new bins, until all demands are fulfilled. For the sake of clarity, we use in this section the term “width” for “weight”.

Let  $W = \{w_1, w_2, \dots, w_m\}$  be the set of item widths, and  $R$  the set of all possible residual widths, computed by subtracting from the bin capacity  $c$  all feasible combinations of item widths (including the empty combination). The level of demand for a certain width  $q$  is

$$L_q = \begin{cases} d_i & \text{if } q = w_i \text{ for some item type } i; \\ 0 & \text{otherwise.} \end{cases} \quad (q \in W \cup R). \quad (18)$$

Additionally, for each  $q \in W \cup R$ , let

- $A(q) = \{p \in R : p > q\}$  if  $q \in W$  (and  $A(q) = \emptyset$  otherwise) denote the set of piece widths that can be used for producing a left piece (item) of width  $q$ ,
- $B(q) = \{p \in W : p + q \in R\}$  denote the set of item widths that, if cut as a left piece, would leave a right piece (residual) of width  $q$ , and
- $C(q) = \{p \in W : p < q\}$  denote the set of item widths that can be cut, as a left piece, from a residual of width  $q$ .

By introducing an integer variable  $x_{pq}$ , that gives the number of times a bin, or a residual of width  $p$ , is cut into a left piece of width  $q$  and a right piece of width  $p - q$  ( $p \in R, q \in$

$W, p > q$ ), the *One-cut* model can be defined as the ILP

$$\min \sum_{q \in W} x_{cq} \quad (19)$$

$$\text{s.t.} \quad \sum_{p \in A(q)} x_{pq} + \sum_{p \in B(q)} x_{p+q,p} \geq L_q + \sum_{r \in C(q)} x_{qr} \quad q \in W \cup R, \quad (20)$$

$$x_{pq} \geq 0 \text{ and integer} \quad p \in R, q \in W, p > q. \quad (21)$$

The objective function (19) minimizes the number of times an item is cut from a bin. Constraints (20) impose that, for each width  $q$ , the sum of the left pieces of width  $q$  plus the sum of the right pieces of width  $q$  is not smaller than the level of demand of width  $q$  plus the number of times a residual of width  $q$  is used to produce smaller items.

**Example 1** (resumed) *For the CSP instance we have  $W = \{2, 3, 4\}$  and  $R = \{1, 2, 3, 4, 5, 6, 7, 9\}$ . We obtain:*

- $A(2) = \{3, 4, 5, 6, 7, 9\}$ ,  $A(3) = \{4, 5, 6, 7, 9\}$ ,  $A(4) = \{5, 6, 7, 9\}$ ,  $A(1) = A(5) = A(6) = A(7) = A(9) = \emptyset$ ;
- $B(1) = B(2) = B(3) = \{2, 3, 4\}$ ,  $B(4) = \{2, 3\}$ ,  $B(5) = \{2, 4\}$ ,  $B(6) = \{3\}$ ,  $B(7) = \{2\}$ ,  $B(9) = \emptyset$ ;
- $C(1) = C(2) = \emptyset$ ,  $C(3) = \{2\}$ ,  $C(4) = \{2, 3\}$ ,  $C(5) = C(6) = C(7) = C(9) = \{2, 3, 4\}$ .

*An optimal solution is then given by  $x_{9,4} = 2$ ,  $x_{5,3} = 2$ , and  $x_{pq} = 0$  otherwise. In other words, we cut two items of width 4 from two bins, and two items of width 3 from the two residuals we have obtained. The two resulting residuals provide two items of width 2.  $\square$*

Set  $R$  can be obtained by running a standard dynamic programming algorithm, or a recursive algorithm, that generates all possible item combinations.

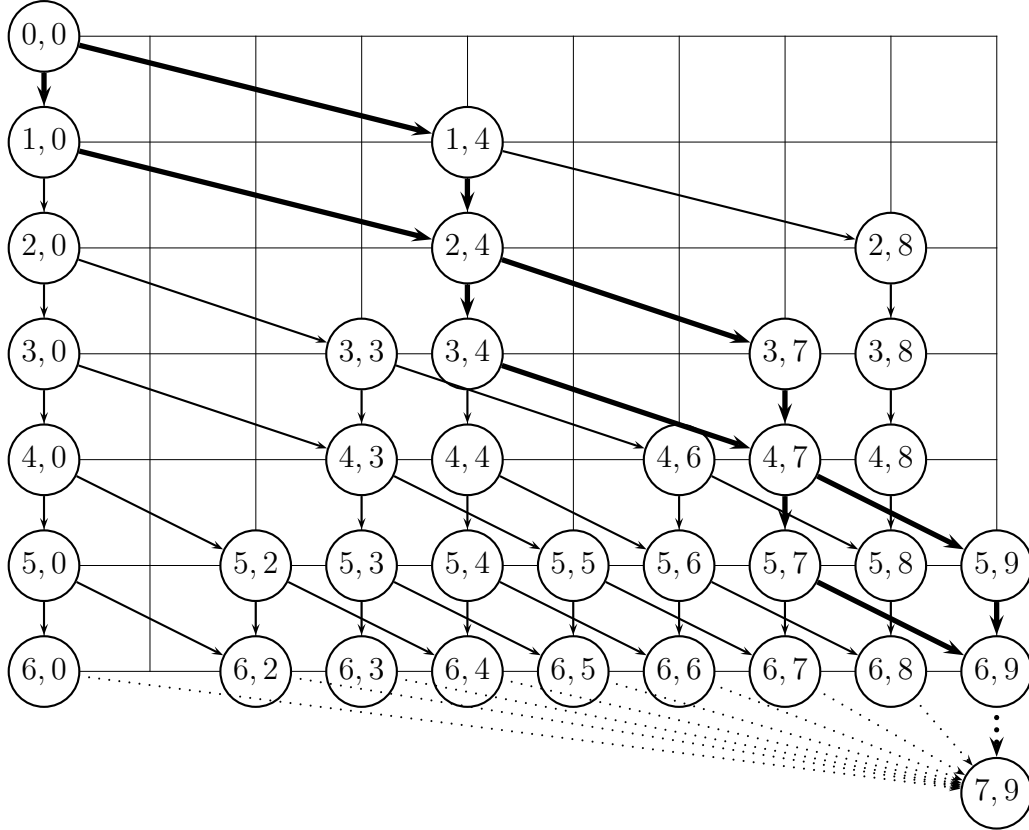
Stadtler [136] studied the combinatorial structure of the one-cut model and extended it by including additional variables and constraints. He also worked on comparing the model and the classical column generation approach, and concluded that “The set of real world cutting stock problems solvable by the one-cut model is only a subset of those which could be tackled by the column generation approach”.

## 3.2 DP-flow Formulation

A simple pseudo-polynomial model is obtained by associating variables to the decisions taken in a classical *dynamic programming* (DP) table. In the BPP model proposed by Cambazard and O’Sullivan [21], known as *DP-flow*, the DP states are represented by a graph in which a path that starts from an initial node and ends at a terminal node represents a feasible filling of a bin. Let us denote by  $(j, d)$  ( $j = 0, \dots, n$  and  $d = 0, \dots, c$ ) a DP state in which decisions have been taken up to item  $j$  and result in a *partial bin filling* of  $d$  units. Let us also denote by  $((j, d), (j + 1, e))$  an arc connecting states  $(j, d)$  and  $(j + 1, e)$ . Such arc expresses the decision on whether packing or not item  $j + 1$  starting from the current state  $(j, d)$ : the state reached by the arc is  $(j + 1, d + w_{j+1})$  if item  $j + 1$  is packed, and  $(j + 1, d)$  otherwise.

**Example 1** (resumed) *The DP table associated with our instance is shown in Figure 2, where states are represented by nodes and organized in  $n + 1$  horizontal layers. The table includes an additional terminal state  $(n + 1, c)$ , and states in layer  $n$  are connected to it by loss arcs (dashed lines), that express the amount of unused capacity in a given bin.  $\square$*

**Figure 2** DP-flow graph construction for Example 1



Let  $A$  denote the set of all arcs. As a feasible bin filling is represented by a path that starts from node  $(0, 0)$  and ends at node  $(n + 1, c)$ , the BPP is to select the minimum number of paths that contain all items. To formulate this decision problem, let us associate an integer variable  $x_{j,d,j+1,e}$  to arc  $((j, d), (j + 1, e)) \in A$ , representing the number of times the arc has been chosen to form paths. Let  $\delta^-((j, d))$  (resp.  $\delta^+((j, d))$ ) denote the set of arcs entering (resp. emanating from) state  $(j, d)$ . The BPP can be then modeled as

$$\min \quad z \tag{22}$$

$$\text{s.t.} \quad \sum_{((j,d),(j+1,e)) \in \delta^+((j,d))} x_{j,d,j+1,e} - \sum_{((j-1,e),(j,d)) \in \delta^-((j,d))} x_{j-1,e,j,d} = \begin{cases} z & \text{if } (j, d) = (0, 0); \\ -z & \text{if } (j, d) = (n + 1, c); \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

$$\sum_{((j-1,d),(j,d+w_j)) \in A} x_{j-1,d,j,d+w_j} = 1 \quad (j = 1, \dots, n), \tag{24}$$

$$x_{j,d,j+1,e} \geq 0 \text{ and integer} \quad ((j, d), (j + 1, e)) \in A. \tag{25}$$

The objective function (22) minimizes the number of bins. Constraints (23) impose the flow (number of bins) conservation at all nodes, while constraints (24) ensure that each item is packed exactly once. Note that a “ $\geq$ ” sign could be used in (24) without affecting the correctness of the model.

**Example 1** (resumed) *For the BPP instance an optimal solution is produced by the two paths highlighted in Figure 2, namely  $[(0,0), (1,4), (2,4), (3,7), (4,7), (5,9), (6,9), (7,9)]$  and  $[(0,0), (1,0), (2,4), (3,4), (4,7), (5,7), (6,9), (7,9)]$ .  $\square$*

Model (22)-(25) has  $O(nc)$  variables and constraints. This formulation is especially tailored for the BPP and is not efficiently extendable to the CSP, because copies of the same item type cannot be grouped together in the DP table. The formulations introduced in the next section overcome this drawback by modeling the CSP instead.

### 3.3 Arc-flow Formulation

An effective CSP pseudo-polynomial formulation, denoted *Arc-flow*, was presented by Valério de Carvalho [142], who used it in his branch-and-price algorithm (see Section 5). To make its comprehension easier, consider again Example 1, and the DP representation depicted in Figure 2. Now imagine that the graph is vertically shrunk, by grouping all states with the same partial bin filling into a single one. In this way, the “vertical” arcs disappear, while the “slanting” ones that connect the same pair of nodes merge into a single arc. Figure 3 shows the counterpart of Figure 2. Note that the loss arcs, which imply no bin filling variation, connect here consecutive nodes instead of (equivalently) going to the terminal node. Let  $A'$  denote the resulting arc set, and  $x_{de}$  the number of times arc  $(d, e) \in A'$  is chosen. The filling of a single bin corresponds to a path from node 0 to node  $c$  in this graph. The CSP can then be modeled as the following ILP:

$$\min \quad z \tag{26}$$

$$\text{s.t.} \quad - \sum_{(d,e) \in \delta^-(e)} x_{de} + \sum_{(e,f) \in \delta^+(e)} x_{ef} = \begin{cases} z & \text{if } e = 0; \\ -z & \text{for } e = c; \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

$$\sum_{(d,d+w_i) \in A'} x_{d,d+w_i} \geq b_i \quad (i = 1, \dots, m), \tag{28}$$

$$x_{de} \geq 0 \text{ and integer} \quad (d, e) \in A', \tag{29}$$

where  $\delta^-(e)$  (resp.  $\delta^+(e)$ ) denotes the set of arcs entering (resp. emanating from)  $e$ .

Constraints (27) impose the flow conservation at all nodes. Constraints (28) impose that, for each item type  $i$ , at least  $b_i$  arcs of length  $w_i$  are used, i.e., that at least  $b_i$  copies of  $i$  are packed.

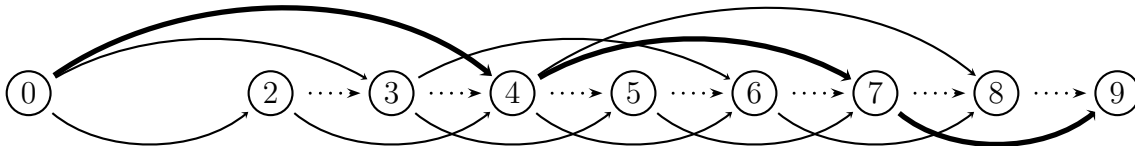
**Example 1** (resumed) *An optimal solution to the CSP instance consists of two identical paths  $[0, 4, 7, 9]$ , highlighted in Figure 3.  $\square$*

Valério de Carvalho [142] additionally proposed a number of improvements to the above basic model, aimed at reducing the number of arcs. For example (see again Figure 3), it is

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**Figure 3** Arc-flow representation of the graph of Figure 2

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enough to only create nodes that correspond to feasible combinations of item weights. In addition, it is proved in [142] that the linear programming relaxation of (26)-(29) has the same solution value as the Gilmore and Gomory [67] model (see Section 5).

In an unpublished report, Brandão and Pedroso [18] proposed an alternative CSP arc-flow formulation, that works on a multi-graph and produces a very compact shrunk representation of the DP states. The corresponding code (see Section 6) appears to be very efficient on benchmark instances.

## 4 Branch-and-Bound

The first attempts to exactly solve the BPP and the CSP were developed in the fifties and in the sixties using LP relaxations and dynamic programming (see Eisemann [51] and Gilmore and Gomory [67, 68, 69]). Starting from the early seventies, research in this field focused on branch-and-bound. To the best of our knowledge, the first branch-and-bound algorithm for the BPP was proposed by Elion and Christofides [50], who adapted the general enumerative scheme proposed by Balas [5] for solving LPs with zero-one variables. Their algorithm produces a binary decision tree in which a node generates two descendant nodes by assigning a certain item to a certain bin, or by excluding it from that bin. The process is initialized by the heuristic solution produced by the BFD algorithm (see Section 2.1) followed by a reshuffle routine. Lower bounds are obtained from a standard LP relaxation. The algorithm could only solve instances of very moderate size.

Later on, thanks to the development of better heuristics, improved lower bounds, and reduction procedures, a more powerful branch-and-bound algorithm, called MTP, was developed by Martello and Toth [105]. During the nineties, this algorithm, whose Fortran code was available, has been the standard reference for the exact solution of the BPP. Their reduction procedures are based on the following dominance criterion. Given an instance  $I$  of the BPP, define a *feasible set*  $F$  as a set of items such that  $\sum_{j \in F} w_j \leq c$ . A feasible set  $F_1$  *dominates* another feasible set  $F_2$  if the optimal solution obtained by imposing  $F_1$  as the content of a bin is not greater than that obtained by imposing  $F_2$  as the content of a bin. Martello and Toth [104] proved the following

**Property 3** *Given two distinct feasible sets  $F_1$  and  $F_2$ , if there exists a partition  $P = \{P_1, \dots, P_\ell\}$  of  $F_2$ , and a subset  $\{j_1, \dots, j_\ell\}$  of  $F_1$  such that  $w_{j_h} \geq \sum_{k \in P_h} w_k$  for  $h = 1, \dots, \ell$ , then  $F_1$  dominates  $F_2$ .*

Clearly, if a feasible set  $F$  containing an item  $j$  dominates all other feasible sets containing the same item  $j$ , then we can impose  $F$  to a bin and reduce the instance accordingly. Checking

all such sets is computationally too heavy, and hence the Martello-Toth *reduction procedure* MTRP limits the search to feasible sets of cardinality at most three and has  $O(n^2)$  time complexity. The procedure was also used (iteratively) to produce, in  $O(n^3)$  time, an improved lower bound  $L_3$ . Algorithm MTP sorts the items according to non-increasing weight, and indexes the bins according to the order in which they are initialized: at each decision node, the next free item is assigned, in turn, to all initialized bins that can accommodate it, and to a new bin. The branch-decision tree is searched according to a depth-first strategy.

Some years after the development of MTP, Scholl, Klein, and Jürgens [128] proposed the other most successful branch-and-bound algorithm for the BPP, known as BISON. They adopted some of the most powerful tools from MTP, and added new lower bounds and emerging techniques like Tabu search, obtaining an improved exact method for the BPP. A couple of years later, Schwerin and Wäscher [130] updated MTP through a lower bound provided by the column generation method of Gilmore and Gomory [67] (see Section 5) in order to improve the competitiveness of MTP with respect to BISON.

In the early noughties Mukhacheva, Belov, Kartack, and Mukhacheva [106] proposed a pattern oriented branch-and-bound algorithm, while Korf [87, 88] proposed a “bin completion” algorithm in which decision nodes are produced by assigning a feasible set to a bin. However, starting from the late nineties, branch-and-price (see Section 5) proved to be very effective, and became the most popular choice for the exact solution of the BPP.

Tree search enumeration is also an ingredient of constraint programming approaches, that are briefly examined in the next section.

## 4.1 Constraint Programming Approaches

In the last decade some attempts have been proposed to solve the BPP through *Constraint Programming* (CP). Shaw [131] presented a new dedicated constraint (later on implemented in the CP optimizer of CPLEX as `IloPack`) based on a set of pruning and propagation rules that also make use of lower bound  $L_2$ . In the following years, some improvements on Shaw’s constraint were proposed. Cambazard and O’Sullivan [21] integrated pseudo-polynomial formulations discussed in Section 3 within the CP approach by Shaw. Dupuis, Schaus, and Deville [46] used lower bound  $L_{2LLM}$  by Labbé, Laporte, and Mercure [89] and an additional reduction algorithm. Schaus, Régim, Van Schaeren, Dullaert, and Raa [122] introduced a filtering rule based on cardinality considerations.

# 5 Branch-and-Price

The *Branch-and-Price* algorithms for the BPP and the CSP are based on the seminal work by Gilmore and Gomory [67, 68], that presented the classical set covering formulation for the CSP, and showed how to solve its continuous relaxation by means of a *column generation* approach.

## 5.1 Column generation

The set covering formulation is based on the enumeration of all *patterns*, i.e., of all combinations of items that can fit into a bin. For the sake of conciseness, in the following we use



$p$  to define both a pattern and its index, and  $P$  to define both the set of patterns and the set of patterns indices.

For the CSP, a *pattern*  $p$  is described by an integer array  $(a_{1p}, a_{2p}, \dots, a_{mp})$ , where  $a_{jp}$  gives the number of copies of item  $j$  that are contained in pattern  $p$ , and satisfies

$$\sum_{j=1}^m a_{jp} w_j \leq c, \text{ and} \quad (30)$$

$$a_{jp} \geq 0 \text{ and integer} \quad (j = 1, \dots, m). \quad (31)$$

Let us introduce an integer variable  $y_p$  that gives, for each  $p \in P$ , the number of times pattern  $p$  is used. Remind that  $d_j$  is the demand of item type  $j$ . The *set covering* formulation of the CSP is given by the ILP

$$\min \sum_{p \in P} y_p \quad (32)$$

$$\text{s.t.} \quad \sum_{p \in P} a_{jp} y_p \geq d_j \quad (j = 1, \dots, m), \quad (33)$$

$$y_p \geq 0 \text{ and integer} \quad (p \in P). \quad (34)$$

Objective function (32) requires the minimization of the number of bins, whereas constraints (33) impose that the subset of selected patterns contains at least  $d_j$  copies of each item  $j$ .

Similarly, for the BPP: (i) a pattern  $p$  is defined by a binary array  $(a_{1p}, a_{2p}, \dots, a_{np})$ , where  $a_{jp}$  is equal to 1 if item  $j$  is contained in pattern  $p$  and 0 otherwise; (ii)  $y_p$  is a decision variable taking the value 1 iff pattern  $p$  is used in the solution. The set covering formulation becomes then

$$\min \sum_{p \in P} y_p \quad (35)$$

$$\text{s.t.} \quad \sum_{p \in P} a_{jp} y_p \geq 1 \quad (j = 1, \dots, n), \quad (36)$$

$$y_p \in \{0, 1\} \quad (p \in P). \quad (37)$$

In both formulations the number of feasible patterns is exponential in the number of items, so enumerating all of them is prohibitive even for moderate-size instances. *Column generation* techniques are consequently adopted. Let us briefly describe the basic technique for the CSP. We first define the continuous relaxation of (32)-(34) by removing the integrality constraints, and heuristically initialize it with a reduced set of patterns  $P' \subseteq P$  that provides a feasible solution. The resulting optimization problem, called the *restricted master problem* (RMP), is

$$\min \sum_{p \in P'} y_p \quad (38)$$

$$\text{s.t.} \quad \sum_{p \in P'} a_{jp} y_p \geq d_j \quad (j = 1, \dots, m), \quad (39)$$

$$y_p \geq 0 \quad (p \in P'). \quad (40)$$

After solving the RMP, let  $\pi_j$  be the dual variable associated with the  $j$ th constraint (39). The existence of a column  $p \notin P'$  that could reduce the objective function value (*pricing problem*) is determined by the *reduced costs*  $\bar{c}_p = 1 - \sum_{j=1}^m a_{jp}\pi_j$  ( $p \notin P'$ ). The column with the most negative reduced cost may be determined by solving an unbounded *knapsack problem* in which the profits are given by the dual variables  $\pi_j$ , i.e., the *slave problem* (SP):

$$\max \sum_{j=1}^m \pi_j v_j, \quad (41)$$

$$\sum_{j=1}^m w_j v_j \leq c, \quad (42)$$

$$v_j \geq 0 \text{ and integer} \quad (j = 1, \dots, m), \quad (43)$$

where  $v_j$  is the number of times item type  $j$  is used. If the solution to the SP has value greater than 1, then the corresponding column (i.e., the corresponding pattern) has negative reduced cost and it is added to the RMP. The process is iterated until no column with negative reduced cost is found, thus providing the optimal solution value to the continuous relaxation of the set covering formulation.

We finally observe that a pattern, as defined by (30)-(31), could contain more than  $d_j$  copies of an item  $j$ , and hence an equivalent definition (*proper pattern*) can be obtained by adding

$$a_{jp} \leq d_j \quad (j = 1, \dots, m). \quad (44)$$

If this formulation is used, the SP consists of a *bounded* knapsack problem, defined by (41)-(43), and

$$v_j \leq d_j \quad (j = 1, \dots, m). \quad (45)$$

This results in a (slightly) stronger lower bound, as the number of feasible patterns becomes smaller. Thorough discussions on this issue may be found in Nitsche, Scheithauer, and Terno [108], and Caprara and Monaci [23]. The lower bound produced by the continuous relaxation is usually very tight, and has been extensively studied in the literature, both from a theoretical and a practical point of view. These studies are presented in the next section.

Alternative column generation approaches make use of the *set partitioning* formulation, in which there is an '=' sign instead of ' $\geq$ ' in constraint (33). Indeed, if an optimal solution to model (32)-(34) contains more than  $d_j$  copies of an item  $j$ , then an equivalent solution can be obtained by arbitrarily removing excess copies from the bins. Consequently the set covering and the set partitioning formulations for the CSP lead to the same optimal solution value. Similar considerations hold for the BPP.

Valério de Carvalho [144] proposed dual cuts to accelerate the column generation process for the CSP. The idea is to add to the RMP "extra" columns (cuts in the dual) that can be found in a fast way and can accelerate the convergence to the continuous optimal solution by reducing the number of "standard" columns generated by the SP. This line of research was pursued by Ben Amor, Desrosiers, and Valério de Carvalho [3], who used dual constraints that are satisfied by at least one optimal dual solution to reduce the typical long-tail effect

of column generation. Clautiaux, Alves, Valério de Carvalho, and Rietz [29] introduced additional dual cuts, as well as a method to tighten lower and upper bounds on the dual variables, in order to stabilize the column generation approach.

Most of the above methods have been used as a basis to produce effective branch-and-price algorithms, that we survey in Section 5.3.

We conclude this section by mentioning some variants of column generation. Briant, Lemaréchal, Meurdesoif, Michel, Perrot, and Vanderbeck [19] compared bundle methods and column generation for solving the LP relaxation of the set covering model, testing them on some BPP and CSP instances. Kiwiel [86] proposed a special bundle method that allows inaccurate solutions to the SP, paired with a rounding heuristic to produce a feasible solution, and experimented it on the CSP. Elhedhli and Gzara [54] recently proposed another heuristic approach to the BPP, based on Lagrangian relaxation and column generation.

## 5.2 Integer Round-Up Property

Let  $L_{LP}$  be the solution value of the continuous relaxation of the set covering formulation, and  $z_{opt}$  the optimal solution value. A BPP (or a CSP) instance is said to have the *Integer Round-Up Property* (IRUP) if the rounded up value of  $L_{LP}$ ,  $\lceil L_{LP} \rceil$ , is equal to  $z_{opt}$ . We call such an instance an *IRUP instance*. On the basis of early computational experiments, it was conjectured in the seventies that the IRUP held for any BPP and CSP instance.

The IRUP conjecture was only proved for some special classes of instances (see, e.g., Berge and Johnson [14], and Marcotte [102]), until it was disproved in the eighties. Marcotte [103] provided an instance for which the IRUP does not hold (*Non-IRUP instance* in the following) with  $n = 24$  and  $c = 3\,397\,386\,255$ . Later on, Chan, Simchi-Levi, and Branel [24] presented a smaller disproving instance, with  $n = 15$  and  $c = 1\,111\,139$ . For both instances the gap between the rounded up lower bound and the optimal solution is exactly one bin. It was then conjectured (see, e.g., Scheithauer and Terno [124, 125]) that  $z_{opt} - \lceil L_{LP} \rceil \leq 1$  holds for any BPP and CSP instance (*Modified Integer Round-Up Property*, MIRUP).

To the best of our knowledge the MIRUP conjecture is still open both for the BPP and the CSP, but a number of interesting results have been obtained while attempting to close it. Kartak [83] presented sufficient conditions under which an instance is Non-IRUP, as well as an algorithm to check them. By performing a huge number of computational tests on randomly generated instances, Schoenfeld [127] (see also Belov and Scheithauer [11]) created a set hard instances, including some satisfying  $z_{opt} - L_{LP} > 1$ . Rietz and Demepe [114] presented methods to construct Non-IRUP instances through perturbations of the item weights that make certain cutting patterns infeasible. Caprara, Díaz Díaz, Dell’Amico, Iori, and Rizzi [22] produced a large set of Non-IRUP instances by using a relationship between the BPP and the edge coloring problem. The smallest such instances have  $n = 13$  and  $c = 100$ , showing that Non-IRUP instances may also appear in practical contexts. They also gave a method to transform an IRUP instance into a Non-IRUP one. Very recently, Kartak, Ripatti, Scheithauer, and Kurz [84] generated classes of Non-IRUP instances through an enumerative method, and showed that the IRUP holds when  $n \leq 9$ . Furthermore, they produced Non-IRUP instances with 10 items. Eisenbrand, Pálvölgyi, and Rothvoß [52] and Newman, Neiman, and Nikolov [107] studied the relationship between the MIRUP conjecture for the BPP and the Becks three-permutation conjecture for discrepancy theory (see Beck

and Sós [8]).

For the CSP, the MIRUP conjecture is an open issue both for the case where (44) is imposed and the case where it is not. For the latter case it is much easier to find Non-IRUP instances, because, as previously mentioned, the resulting lower bound is weaker.

We conclude by observing that all Non-IRUP instances of the literature have been solved exactly. In Section 6.1 we discuss a method to generate Non-IRUP instances that are very difficult to solve exactly.

### 5.3 Branch(-and-Cut)-and-Price Algorithms

When the solution obtained at the end of the column generation method of Section 5.1 is fractional, an additional effort is required to find a feasible integer solution. The generation of all possible patterns followed by the direct solution of (32)-(34) at integrality is the most obvious option, but it can only be adopted for instances of small size, or characterized by a special structure (see, e.g., Goulimis [71]). Other, non exact, methods simply use rounding heuristics (like, e.g., Roodman [117], Haessler and Sweeney [75], and Holthaus [76]), but their efficiency strongly depends on the instances at hand. When these methods fail in producing an optimal integer solution, one can embed the column generation lower bound  $L_{LP}$  into an enumeration tree, thus obtaining a *branch-and-price* algorithm. The main difficulty of this approach is that the branching decisions that have been taken during the enumeration must be embedded in the master and/or the slave problem, so as to avoid the generation of columns that have been excluded by the branch decisions. We review in this section the main methods that have been proposed in the literature to handle this issue.

The first branch-and-price algorithm for the BPP is probably the one proposed by Vance, Barnhart, Johnson, and Nemhauser [146] in 1994. At each decision node the algorithm considers those bins for which the decision variable  $y_p$  is fractional, and selects the pair of items that are fractionally packed into the same bin and have largest total weight: following a branching rule originally developed by Ryan and Foster [121] for set partitioning problems, such items are then forced to be packed either together or separately. In the former case the resulting subproblem is a knapsack problem with an additional constraint, while in the latter case it is sufficient to merge the two items into a unique one. Early termination of nodes without performing the complete column generation is obtained by using a bound due to Farley [57]. The following year Scheithauer and Terno [123] proposed a hybrid strategy for the CSP, oriented to the conjecture that the MIRUP holds for the instance at hand. They first reduce the instance by solving its continuous relaxation and rounding down the solution, so as to obtain a partial integer solution and a residual instance. The residual instance is then attacked through heuristics algorithms and, if they fail in producing an overall optimal solution, it is exactly solved through a branch-and-bound algorithm which includes pricing ideas.

A few years later, Vance [145] focused on the CSP and showed that the application of the classical Dantzig–Wolfe [39] decomposition to the CSP model (9)-(13) leads to the set covering model (32)-(34), and used this result to implement two specifically tailored branching rules.

In the late nineties, Valério de Carvalho [142] proposed a column generation approach based on the arc-flow formulation of Section 3.3, which branches on a fractional flow variable

$x$  by imposing it to be either not smaller than  $\lceil x \rceil$  or not greater than  $\lfloor x \rfloor$ . The branching constraints are directly added to the master. The slave is in this case a standard knapsack problem, which is solved through dynamic programming.

Vanderbeck [147] introduced a branch-and-price algorithm whose rule is to branch on a set of columns. This is obtained by adding a constraint to the master to impose that the sum  $s$  of the variables associated with such set are either not smaller than  $\lceil s \rceil$  or not greater than  $\lfloor s \rfloor$ . In this way the descendant nodes involve a complicated variant of knapsack problem. The convergence of the algorithm is improved by cut generation at the decision nodes, so that the method can be seen as a *Branch-and-Cut-and-Price* algorithm. Later on, Vanderbeck [148, 149] tested on BPP and CSP instances some branching schemes he developed for general branch-and-price algorithms.

Degraeve and Schrage [41] proposed a branch-and-price approach to the CSP, which selects for branching a pattern associated with a fractional variable. A specific constraint is added to the slave in order to prevent such pattern to be generated at descendant nodes. Degraeve and Peeters [40] improved the algorithm by adding heuristics, pruning rules, and a sub-gradient procedure to speed up the solution of the LP relaxations. In addition, they adopted an efficient way to handle decision nodes, by focusing on the solution of the sub-problem obtained by subtracting from the item demands the values of the rounded-down LP solution.

In the early noughties, Scheithauer, Terno, Muller, and Belov [126] proposed an exact solution approach for the CSP based on cutting plane generation. The algorithm computes a lower bound by solving the continuous relaxation of the set covering formulation, and an upper bound by using heuristics. If there is a gap between these two values, Chvátal-Gomory cuts [27] are added to the formulation to possibly increase the lower bound value, and the process is iterated. The slave is solved by a specifically tailored branch-and-bound method that takes into account the dual variables associated with the additional constraints. The method was improved in Belov and Scheithauer [10], and then embedded into a branch-and-price algorithm in Belov and Scheithauer [11]. The resulting algorithm directly branches on the variables associated with the patterns, selecting the variable whose fractional value is closer to 0.5. Later on, Belov, Scheithauer, Alves, and Valério de Carvalho [12] investigated the performance of combining Chvátal-Gomory cuts and arc-flow formulations, which however did not prove to be very effective.

The list of papers commented in this section is not exhaustive, as the literature on branch-and-price algorithms for the BPP and the CSP is huge. For further details, we refer the reader to, e.g., the specific survey by Ben Amor and Valério de Carvalho [13], and to Lübbecke and Desrosiers [99], who treat relevant implementation issues, including specific considerations on the BPP and the CSP.

## 6 Experimental Evaluation

One of the purposes of this survey is to experimentally compare the different solution methods in order to evaluate their average efficiency. We performed the experiments on various sets of instances. benchmarks and computer codes are available in a dedicated web page.

## 6.1 Benchmarks

We used 3 different benchmarks: instances previously used in the literature, randomly generated instances, and instances especially designed so that an exact algorithm can hardly prove the optimality of a solution. All instances are downloadable from the web page <http://or.dei.unibo.it/library/bpplib> (referred to in the following as the *BPPLIB*).

### Literature instances

We tested the algorithms on the instances proposed by:

- Falkenauer [55]: two classes of 80 instances each, available at Beasley’s [7] OR library: the first class has uniformly distributed item sizes (‘Falkenauer U’) with  $n$  between 120 and 1000 and  $c = 150$ . The second class (‘Falkenauer T’) includes the so-called *triplets*, i.e., groups of three items (one large, two small) that need to be assigned to the same bin in any optimal packing, with  $n$  between 60 and 501 and  $c = 1000$ ;
- Scholl, Klein, and Jürgens [128]: three sets of 720, 480, and 10, respectively, uniformly distributed instances (from <http://www.wiwi.uni-jena.de/entscheidung/binpp/>) with  $n$  between 50 and 500 and  $c$  between 100 and 150 (set ‘Scholl 1’), equal to 1000 (set ‘Scholl 2’), and equal to 100 000 (set ‘Scholl 3’), respectively;
- Wäscher and Gau [150]: 17 hard instances (‘Wäscher’ in the tables) available at page [http://paginas.fe.up.pt/~esicup/tiki-list\\_file\\_gallery.php?galleryId=1](http://paginas.fe.up.pt/~esicup/tiki-list_file_gallery.php?galleryId=1) (which also hosts the next two sets), with  $n$  between 57 and 239 and  $c = 10\,000$ ;
- Schwerin and Wäscher [129]: two sets (‘Schwerin 1’ and ‘Schwerin 2’) of 100 instances each with  $n = 100$  and  $n = 120$ , respectively, and  $c = 1000$ ;
- Schoenfeld [127]: 28 instances (‘Hard28’) with  $n$  between 160 and 200, and  $c = 1000$ .

### Randomly generated instances

In order to better evaluate the behavior of the exact algorithms with respect to the instance characteristics, we randomly generated BPP instances with different values of

- $n \in \{50, 100, 200, 300, 400, 500, 750, 1000\}$ ,
- $c \in \{50, 75, 100, 120, 125, 150, 200, 300, 400, 500, 750, 1000\}$ ,
- $w_{\min} \in \{0.1c, 0.2c\}$ ,
- $w_{\max} \in \{0.7c, 0.8c\}$ ,

For each quadruplet, 10 instances were obtained by uniformly randomly generating the weights in  $[w_{\min}, w_{\max}]$ , producing in total 3840 instances.

### Difficult instances

As it will be seen in Section 6.3, all the above instances can be solved in less than 10 minutes by at least one of the softwares we tested. In order to test them on more challenging benchmarks, we designed a new class of instances.

The *augmented Non-IRUP* (ANI) instances were derived from a benchmark, called  $B$  in the following, proposed by Caprara, Dell’Amico, Díaz Díaz, Iori, and Rizzi [22]. Benchmark  $B$  (available at [http://www.or.unimore.it/resources/BPP\\_non\\_IRUP/instances.html](http://www.or.unimore.it/resources/BPP_non_IRUP/instances.html)) consists of 15-item BPP Non-IRUP instances satisfying  $\sum_{j=1}^{15} w_j = L_{LP} c$  (see Section 5.2), for which  $L_{LP} = 3$  and the optimal solution has value 4. An augmented Non-IRUP instance was obtained from an instance of  $B$  by iteratively adding to it a triplet of items such that: (i) their total weight equals  $c$ , and (ii) for at least one of them, say having weight  $w_k$ , there is no subset  $S$  of items currently in the instance such that  $w_k + \sum_{j \in S} w_j = c$ . Whenever (ii) could not be satisfied for the current triplet, both the current capacity and the weights generated so far were doubled. We generated five sets of 50 ANI instances each, with  $n \in \{201, 402, 600, 801, 1002\}$  (remind that  $n$  must be a multiple of 3). Concerning the capacities, it was imposed to the five sets that the value of  $c$  could not exceed an upper bound  $\bar{c}$ , respectively equal to 2 500, 10 000, 20 000, 40 000, and 80 000. Whenever this could not be ensured, the instance was discarded and a new instance was generated.

It is necessary to clarify in which sense the above ANI instances are *difficult* to solve exactly. Exact algorithms (or specifically tailored heuristics or metaheuristics) can indeed find an optimal solution, but they struggle with proving its optimality, as the solution value is higher than lower bound  $L_{LP}$ . For the sake of comparison, we also generated five sets of “easier” *augmented IRUP* (AI) instances with  $n + 1$  items, obtained from the ANI ones by splitting one of the 15 original items into two items so that the resulting 16 items fit into 3 bins, i.e., the Non-IRUP is lost. For the AI instances, all bins are completely filled, so the continuous relaxation provides the optimal solution value, and the only difficulty is to construct a feasible solution having the same value.

## 6.2 Computer codes

We computationally evaluated, among the solution methods treated in the previous sections, all those for which the corresponding source code is available online, plus the classical Pascal code of Bison, that was kindly provided by the authors. In addition, we included a number of methods for which the computer code can be easily implemented. The best computer codes are either linked or downloadable from the BPPLIB.

We tested the following computer codes for the optimal solution of the BPP:

- Branch-and-bound (see Section 4):
  - **MTP**, Fortran code by Martello and Toth [105];
  - **BISON**, Pascal code by Scholl, Klein, and Jürgens [128], kindly provided by Armin Scholl;
  - **CVRPSEP**, C code by J. Lysgaard, included in a package, CVRPSEP, as a part of a separation routine for the Capacitated Vehicle Routing Problem (see [101]). The code has been produced using the procedures of MTP. The whole package is available at <http://www.hha.dk/lys/CVRPSEP.htm>.
- Branch-and-price (see Section 5.3):

- **VANCE**, C++ implementation of the algorithm by Vance, Barnhart, Johnson, and Nemhauser [146], using CPLEX 12.6.0 for the LP relaxations and the knapsack problems with additional constraints;
  - **BELOV**, C++ implementation by Belov of the algorithm by Belov and Scheithauer [11], using CPLEX 12.6.0 for the inner routines, available at web page <http://www.math.tu-dresden.de/~capad/cpd-sw.html>;
  - **SCIP-BP**, freeware SCIP 3.0.2 C code for a branch-and-price BPP algorithm, available at <http://scip.zib.de/doc/examples/Binpacking/BRANCHING.php>, that uses the Ryan and Foster [121] branching rule, also adopted in VANCE.
- Pseudo-polynomial formulations (see Section 3):
    - **ONECUT**, C++ implementation of the one-cut model by Dyckhoff [47] and Stadtler [136], using CPLEX 12.6.0 for the ILP, available at the BPPLIB;
    - **ARCFLOW**, C++ implementation of the arc-flow model by Valério de Carvalho [142], using CPLEX 12.6.0 for the ILP, available at the BPPLIB;
    - **DPFLOW**, C++ implementation of the DP-flow model by Cambazard and O’Sullivan [21], using CPLEX 12.6.0 for the ILP, available at the BPPLIB;
    - **VPSOLVER**, C++ implementation by Brandão and Pedroso [18], which uses Gurobi 5.6 as inner routine, available at <https://code.google.com/p/vpsolver>.
  - Other methods:
    - **BASIC ILP**, C++ implementation of the introductory ILP model (3)-(7), implemented using CPLEX 12.6.0;
    - **CSTRPROG**, C++ implementation of an algorithm based on constraint programming (see Section 4.1), using the CP optimizer of CPLEX 12.6.0 and selecting constraint `IloPack` (see Shaw [131]), and a search phase based on an FFD strategy.

We preliminary computed lower and upper bounds through a simple procedure, **BFDL2**, which includes approximation algorithm BFD of Section 2.1 and lower bound  $L_2$  of Section 2.2. The computer codes were only executed on instances for which lower and upper bound did not coincide. For our C++ implementations, the BFD upper bound was passed to CPLEX.

## 6.3 Experiments

All the experiments but those in Table 13 were executed on an Intel Xeon 3.10 GHz with 8 GB RAM, equipped with four cores. In order to allow fair comparisons with other algorithms and machines, all our experiments were performed with a single core, and the number of threads was set to one for all solvers.

Tables 1-3 give the results for the literature instances. Table 1 provides the results obtained by running the codes with a time limit of one minute. Columns 1 and 2 identify the benchmark and give the number of instances for which the codes were executed. The



Table 1: Number of literature instances (average gap wrt lower bound) solved in less than one minute

Set	tested inst.	Branch-and-bound			Branch-and-price			Pseudo-polynomial				Others	
		MTP	BISON	CVRPSEP	VANCE	BELOV	SCIP-BP	ONECUT	ARCFLOW	DPFLOW	VPSOLVER	BASIC ILP	CSTRPROG
Falkenauer U	74	22 (1.7)	44 (0.4)	22 (1.8)	53 (1.2)	<b>74</b> (0.0)	18 (2.1)	<b>74</b> (0.0)	<b>74</b> (0.0)	37 (1.8)	<b>74</b> (0.0)	10 (2.3)	28 (2.0)
Falkenauer T	80	6 (7.0)	42 (0.5)	0 (11.0)	76 (0.1)	57 (0.3)	35 (4.5)	<b>80</b> (0.0)	<b>80</b> (0.0)	40 (8.8)	<b>80</b> (0.0)	7 (7.0)	39 (8.8)
Scholl 1	323	242 (0.3)	288 (0.1)	223 (0.3)	<b>323</b> (0.0)	<b>323</b> (0.0)	244 (0.2)	<b>323</b> (0.0)	<b>323</b> (0.0)	289 (0.1)	<b>323</b> (0.0)	212 (0.3)	90 (0.6)
Scholl 2	244	130 (0.6)	233 (0.0)	65 (1.4)	204 (0.2)	<b>244</b> (0.0)	67 (1.2)	118 (0.4)	202 (0.1)	58 (1.3)	208 (0.1)	90 (1.0)	122 (1.3)
Scholl 3	10	0 (1.5)	3 (0.7)	0 (4.1)	<b>10</b> (0.0)	<b>10</b> (0.0)	0 (4.1)	0 (4.1)	0 (4.1)	0 (4.1)	<b>10</b> (0.0)	0 (2.7)	0 (4.1)
Wäscher	17	0 (1.0)	10 (0.4)	0 (1.0)	6 (0.6)	<b>17</b> (0.0)	0 (1.0)	0 (1.0)	0 (1.0)	0 (1.0)	6 (0.6)	4 (0.8)	7 (0.6)
Schwerin 1	100	15 (0.9)	<b>100</b> (0.0)	9 (0.9)	<b>100</b> (0.0)	<b>100</b> (0.0)	0 (1.0)	<b>100</b> (0.0)	<b>100</b> (0.0)	0 (1.0)	<b>100</b> (0.0)	32 (0.7)	<b>100</b> (0.0)
Schwerin 2	100	4 (1.4)	63 (0.4)	0 (1.4)	<b>100</b> (0.0)	<b>100</b> (0.0)	0 (1.4)	<b>100</b> (0.0)	<b>100</b> (0.0)	0 (1.4)	<b>100</b> (0.0)	36 (0.7)	60 (0.8)
Hard28	28	0 (1.0)	0 (1.0)	0 (1.0)	11 (0.6)	<b>28</b> (0.0)	7 (0.8)	6 (0.8)	16 (0.4)	0 (1.0)	27 (0.0)	0 (1.0)	0 (1.0)
Total	976	419 (0.9)	783 (0.1)	319 (1.4)	883 (0.1)	953 (0.0)	371 (1.0)	801 (0.2)	895 (0.1)	424 (1.2)	928 (0.0)	391 (1.0)	446 (1.3)

Table 2: Average time in seconds (standard deviation) for solving literature instances

Set	tested inst.	Branch-and-bound			Branch-and-price			Pseudo-polynomial				Others	
		MTP	BISON	CVRPSEP	VANCE	BELOV	SCIP-BP	ONECUT	ARCFLOW	DPFLOW	VPSOLVER	BASIC ILP	CSTRPROG
Falkenauer U	74	42.8 (27.2)	24.5 (29.5)	42.2 (27.6)	24.1 (25.2)	0.0 (0.0)	50.1 (19.1)	0.2 (0.1)	0.2 (0.1)	38.8 (23.9)	0.1 (0.0)	61.4 (41.2)	38.8 (27.9)
Falkenauer T	80	55.5 (15.9)	30.6 (29.3)	60.2 (0.3)	14.8 (19.2)	24.7 (26.8)	39.4 (25.6)	8.7 (10.7)	3.5 (6.8)	41.7 (22.0)	0.4 (0.5)	58.1 (8.9)	34.2 (27.6)
Scholl 1	323	15.1 (26.0)	7.0 (18.8)	19.4 (27.6)	3.6 (7.5)	0.0 (0.0)	22.4 (24.3)	0.1 (0.1)	0.1 (0.3)	13.0 (19.1)	0.1 (0.1)	23.1 (28.3)	44.3 (25.9)
Scholl 2	244	28.2 (29.9)	3.0 (12.7)	44.2 (26.4)	18.6 (24.3)	0.3 (0.4)	49.2 (20.2)	38.7 (25.6)	18.9 (23.1)	50.4 (19.4)	14.0 (21.5)	40.7 (27.2)	31.7 (29.0)
Scholl 3	10	60.0 (0.0)	42.0 (29.0)	60.0 (0.0)	1.9 (0.8)	14.1 (1.5)	60.0 (0.0)	63.9 (3.1)	61.1 (0.3)	60.0 (0.0)	6.3 (3.9)	60.0 (0.0)	60.0 (0.0)
Wäscher	17	60.0 (0.0)	24.7 (30.4)	60.0 (0.0)	52.0 (18.9)	0.1 (0.1)	60.0 (0.1)	60.7 (0.2)	60.5 (0.3)	60.0 (0.0)	49.4 (26.6)	49.9 (19.3)	37.2 (28.4)
Schwerin 1	100	51.1 (21.3)	0.0 (0.0)	55.4 (15.6)	0.3 (0.0)	1.0 (0.3)	60.1 (0.0)	13.1 (9.5)	1.5 (0.6)	59.6 (0.3)	0.3 (0.2)	43.0 (25.8)	4.4 (7.4)
Schwerin 2	100	57.6 (11.8)	22.2 (29.1)	60.0 (0.0)	0.5 (0.1)	1.4 (0.3)	60.1 (0.0)	11.7 (7.8)	1.5 (0.7)	59.6 (0.3)	0.3 (0.1)	43.1 (25.3)	27.1 (27.8)
Hard28	28	60.0 (0.0)	60.0 (0.0)	60.0 (0.0)	48.9 (20.8)	7.3 (11.9)	51.2 (16.8)	54.6 (11.4)	40.6 (20.0)	60.0 (0.0)	14.2 (17.9)	60.0 (0.0)	60.0 (0.0)
Total	976	34.4 (22.8)	12.3 (17.9)	40.8 (19.5)	11.3 (13.0)	2.7 (2.7)	42.2 (17.1)	16.3 (9.5)	8.2 (7.2)	38.9 (14.8)	5.0 (6.5)	39.3 (25.6)	34.6 (24.3)

Table 3: Number of literature instances solved in less than ten minutes

Set	tested inst.	BISON	BELOV	ARCFLOW	VPSOLVER
Falkenauer U	74	50	<b>74</b>	<b>74</b>	<b>74</b>
Falkenauer T	80	47	<b>80</b>	<b>80</b>	<b>80</b>
Scholl 1	323	290	<b>323</b>	<b>323</b>	<b>323</b>
Scholl 2	244	234	<b>244</b>	231	242
Scholl 3	10	3	<b>10</b>	0	<b>10</b>
Wäscher	17	10	<b>17</b>	4	13
Schwerin 1	100	<b>100</b>	<b>100</b>	<b>100</b>	<b>100</b>
Schwerin 2	100	63	<b>100</b>	<b>100</b>	<b>100</b>
Hard28	28	0	<b>28</b>	26	26
Total	976	797	<b>976</b>	938	968

column associated with each code provides the number of such instances that were solved to proven optimality and, in parentheses, the average value of the absolute gap  $g$  between the solution value and the lower bound produced by the code. For the cases where an algorithm could solve all instances, the corresponding number appears in bold. When the time limit is very small, codes BELOV, SCIP-BP, and VPSOLVER can sometimes terminate without producing a decent lower and/or upper bound: in order to avoid anomalous results, the gap was always computed as the minimum between  $g$  and the gap produced by BFDL2. The last line of the table reports the total number of solved instances and, in parentheses, the overall average gap.

Table 2 has the same structure as Table 1 but the entries provide, for each computer code, the average CPU time expressed in seconds and, in parentheses, the corresponding standard deviation. The entries in the last line give in this case the average CPU time and, in parentheses, the standard deviation computed over all instances for which the code was executed. It must be observed that, for the computer codes that invoke CPLEX, SCIP, or Gurobi, the actual CPU time spent on an instance turns out, in some cases, to be greater than the time limit. In most cases the difference was irrelevant but for BASIC ILP. Indeed model (3)-(7) involves a huge number of variables, and CPLEX can get stuck in the cutting plane loop, which needs a high time and cannot be interrupted freely. This explains a couple of average times higher than 60 seconds (in Tables 2 and 7). The instances that required a CPU time much larger than 60 seconds were counted as unsolved by BASIC ILP (while the improvement coming from the additional CPU time spent turned out to be irrelevant).

Tables 1 and 2 show that, for the literature instances,

1. among the (old) branch-and-bound codes, BISON is the only one capable of solving many instances;
2. two branch-and-price algorithms (VANCE and, in particular, BELOV) have satisfactory results, while SCIP-BP does not appear to be competitive. The only difficult instances for BELOV appear to be the Falkenauer T;
3. ARCFLOW and VPSOLVER are the most efficient pseudo-polynomial approaches, and the very old ONECUT method has an overall decent performance;
4. as it could be expected, the efficiency of BASIC ILP and CSTRPROG is quite low.

We selected the winner of each algorithmic class (BISON, BELOV, and VPSOLVER) for an additional round of tests (on the same instances) with a time limit of 10 minutes. By considering that the performance of ARCFLOW is competitive with that of VPSOLVER, and that its graph construction is considerably simpler, we decided to include it in this round. The number of solved instances within the larger time limit are provided in Table 3. Overall, the four algorithms exhibited a satisfactory behavior. In particular BELOV solved all instances and VPSOLVER almost all of them. ARCFLOW and BISON solved 96% and 82% of the instances, respectively. Instances with very large capacity values turned out to be particularly hard for ARCFLOW.

The next group of six tables refers to the randomly generated instances. Tables 4-6 provide the number of instances solved by each computer code (and, in parentheses, the gap), with a time limit of one minute, when varying the items characteristics. In Table 4

Table 4: Number of random instances solved in less than one minute (average gap wrt lower bound) when varying  $n$ 

$n$	tested inst.	Branch-and-bound			Branch-and-price			Pseudo-polynomial				Others	
		MTP	BISON	CVRPSEP	VANCE	BELOW	SCIP-BP	ONECUT	ARCFLOW	DPFLOW	VPSOLVER	BASIC ILP	CSTRPROG
50	165	163 (0.0)	<b>165</b> (0.0)	164 (0.0)	<b>165</b> (0.0)	<b>165</b> (0.0)	<b>165</b> (0.0)	<b>165</b> (0.0)	<b>165</b> (0.0)	<b>165</b> (0.0)	<b>165</b> (0.0)	157 (0.0)	71 (0.7)
100	271	243 (0.1)	257 (0.1)	239 (0.1)	<b>271</b> (0.0)	<b>271</b> (0.0)	<b>271</b> (0.0)	<b>271</b> (0.0)	<b>271</b> (0.0)	<b>271</b> (0.0)	<b>271</b> (0.0)	237 (0.1)	132 (0.6)
200	359	237 (0.4)	290 (0.2)	220 (0.6)	358 (0.0)	<b>359</b> (0.0)	293 (0.2)	358 (0.0)	<b>359</b> (0.0)	292 (0.2)	<b>359</b> (0.0)	201 (0.4)	171 (0.8)
300	393	166 (0.8)	265 (0.3)	144 (1.1)	387 (0.0)	<b>393</b> (0.0)	155 (0.8)	385 (0.0)	391 (0.0)	243 (0.6)	<b>393</b> (0.0)	115 (0.8)	140 (1.2)
400	425	151 (1.1)	244 (0.5)	138 (1.4)	416 (0.0)	<b>425</b> (0.0)	114 (1.1)	408 (0.1)	421 (0.0)	193 (1.1)	<b>425</b> (0.0)	92 (1.0)	104 (1.7)
500	414	121 (1.4)	208 (0.6)	128 (1.6)	394 (0.0)	<b>414</b> (0.0)	69 (1.7)	394 (0.1)	402 (0.0)	169 (1.3)	413 (0.0)	60 (1.5)	61 (2.0)
750	433	93 (2.0)	214 (0.7)	98 (2.3)	99 (2.1)	<b>433</b> (0.0)	22 (2.7)	401 (0.2)	415 (0.1)	120 (2.0)	431 (0.0)	54 (2.5)	23 (2.8)
1000	441	78 (2.6)	196 (0.8)	73 (3.1)	62 (2.8)	<b>441</b> (0.0)	0 (3.6)	407 (0.2)	416 (0.1)	67 (3.1)	434 (0.0)	39 (3.3)	7 (3.6)
Overall	2901	1252 (1.2)	1839 (0.5)	1204 (1.5)	2152 (0.8)	<b>2901</b> (0.0)	1089 (1.5)	2789 (0.1)	2840 (0.0)	1520 (1.2)	2891 (0.0)	955 (1.4)	709 (1.9)

Table 5: Number of random instances solved in less than one minute (average gap wrt lower bound) when varying  $c$ 

$c$	tested inst.	Branch-and-bound			Branch-and-price			Pseudo-polynomial				Others	
		MTP	BISON	CVRPSEP	VANCE	BELOW	SCIP-BP	ONECUT	ARCFLOW	DPFLOW	VPSOLVER	BASIC ILP	CSTRPROG
50	223	125 (0.5)	191 (0.1)	145 (0.6)	162 (0.4)	<b>223</b> (0.0)	86 (0.8)	<b>223</b> (0.0)	<b>223</b> (0.0)	205 (0.1)	<b>223</b> (0.0)	83 (0.8)	73 (0.9)
75	240	137 (0.7)	187 (0.2)	141 (0.8)	176 (0.5)	<b>240</b> (0.0)	96 (1.0)	<b>240</b> (0.0)	<b>240</b> (0.0)	208 (0.2)	<b>240</b> (0.0)	92 (0.9)	84 (1.2)
100	234	111 (0.8)	177 (0.2)	116 (1.0)	172 (0.6)	<b>234</b> (0.0)	89 (1.1)	<b>234</b> (0.0)	<b>234</b> (0.0)	185 (0.3)	<b>234</b> (0.0)	71 (1.0)	64 (1.3)
120	241	110 (0.8)	172 (0.3)	112 (1.0)	181 (0.5)	<b>241</b> (0.0)	91 (1.1)	<b>241</b> (0.0)	<b>241</b> (0.0)	168 (0.5)	<b>241</b> (0.0)	82 (1.0)	66 (1.3)
125	251	127 (0.8)	176 (0.3)	129 (1.0)	192 (0.5)	<b>251</b> (0.0)	101 (1.1)	<b>251</b> (0.0)	<b>251</b> (0.0)	174 (0.6)	<b>251</b> (0.0)	84 (1.1)	77 (1.3)
150	240	101 (0.9)	165 (0.3)	90 (1.2)	181 (0.6)	<b>240</b> (0.0)	95 (1.1)	<b>240</b> (0.0)	<b>240</b> (0.0)	143 (0.8)	<b>240</b> (0.0)	72 (1.1)	57 (1.5)
200	246	95 (1.0)	156 (0.3)	89 (1.2)	184 (0.6)	<b>246</b> (0.0)	99 (1.1)	<b>246</b> (0.0)	<b>246</b> (0.0)	127 (0.9)	<b>246</b> (0.0)	74 (1.1)	53 (1.5)
300	237	86 (1.0)	134 (0.4)	77 (1.2)	172 (0.6)	<b>237</b> (0.0)	80 (1.2)	<b>237</b> (0.0)	<b>237</b> (0.0)	79 (1.3)	<b>237</b> (0.0)	67 (1.2)	59 (1.5)
400	245	96 (1.1)	122 (0.5)	81 (1.4)	184 (0.6)	<b>245</b> (0.0)	95 (1.3)	<b>245</b> (0.0)	<b>245</b> (0.0)	71 (1.5)	<b>245</b> (0.0)	80 (1.2)	51 (1.6)
500	243	90 (1.1)	125 (0.5)	76 (1.4)	179 (0.6)	<b>243</b> (0.0)	77 (1.3)	241 (0.0)	242 (0.0)	56 (1.6)	<b>243</b> (0.0)	79 (1.2)	45 (1.6)
750	249	82 (1.1)	119 (0.6)	70 (1.4)	183 (0.6)	<b>249</b> (0.0)	91 (1.3)	211 (0.2)	229 (0.1)	55 (1.6)	<b>249</b> (0.0)	84 (1.2)	36 (1.7)
1000	252	92 (1.1)	115 (0.6)	78 (1.4)	186 (0.6)	<b>252</b> (0.0)	89 (1.3)	180 (0.5)	212 (0.3)	49 (1.6)	242 (0.0)	87 (1.2)	44 (1.6)
Overall	2901	1252 (0.9)	1839 (0.3)	1204 (1.1)	2152 (0.6)	<b>2901</b> (0.0)	1089 (1.1)	2789 (0.1)	2840 (0.0)	1520 (0.9)	2891 (0.0)	955 (1.1)	709 (1.4)

Table 6: Number of random instances solved in less than one minute (average gap wrt lower bound) when varying weight range

Range	tested inst.	Branch-and-bound			Branch-and-price			Pseudo-polynomial				Others	
		MTP	BISON	CVRPSEP	VANCE	BELOW	SCIP-BP	ONECUT	ARCFLOW	DPFLOW	VPSOLVER	BASIC ILP	CSTRPROG
0.1 / 0.7	785	337 (0.7)	590 (0.2)	385 (0.6)	541 (0.6)	<b>785</b> (0.0)	162 (1.1)	700 (0.2)	737 (0.1)	274 (1.0)	776 (0.0)	106 (1.1)	384 (0.8)
0.1 / 0.8	729	222 (0.7)	339 (0.5)	242 (0.7)	476 (0.5)	<b>729</b> (0.0)	233 (0.9)	703 (0.1)	716 (0.0)	328 (0.8)	728 (0.0)	197 (0.9)	171 (1.0)
0.2 / 0.7	878	229 (2.2)	406 (0.7)	206 (2.9)	646 (1.2)	<b>878</b> (0.0)	340 (2.3)	877 (0.0)	<b>878</b> (0.0)	569 (1.6)	<b>878</b> (0.0)	281 (2.1)	116 (3.0)
0.2 / 0.8	509	464 (0.1)	504 (0.0)	371 (0.3)	489 (0.0)	<b>509</b> (0.0)	354 (0.3)	<b>509</b> (0.0)	<b>509</b> (0.0)	349 (0.3)	<b>509</b> (0.0)	371 (0.2)	38 (0.9)
Overall	2901	1252 (0.9)	1839 (0.3)	1204 (1.1)	2152 (0.6)	<b>2901</b> (0.0)	1089 (1.1)	2789 (0.1)	2840 (0.0)	1520 (0.9)	2891 (0.0)	955 (1.1)	709 (1.4)

Table 7: Average time in seconds (standard dev.) for solving random instances when varying  $n$ 

$n$	tested inst.	Branch-and-bound			Branch-and-price			Pseudo-polynomial				Others	
		MTP	BISON	CVRPSEP	VANCE	BELOV	SCIP-BP	ONECUT	ARCFLOW	DPFLOW	VPSOLVER	BASIC ILP	CSTRPROG
50	165	0.8 (6.6)	0.0 (0.0)	0.4 (4.7)	0.0 (0.1)	0.0 (0.0)	0.9 (0.7)	0.1 (0.3)	0.1 (0.1)	0.5 (0.8)	0.0 (0.0)	4.5 (14.8)	34.8 (29.4)
100	271	7.4 (18.9)	3.8 (13.7)	8.4 (19.7)	0.1 (0.2)	0.0 (0.0)	4.6 (7.1)	0.8 (2.5)	0.3 (0.4)	5.0 (7.5)	0.1 (0.1)	9.4 (20.5)	31.4 (29.6)
200	359	21.6 (28.3)	12.0 (23.7)	25.0 (28.9)	1.1 (4.1)	0.0 (0.0)	22.6 (21.8)	2.4 (7.0)	0.8 (2.6)	21.0 (22.1)	0.3 (0.9)	29.4 (28.8)	33.0 (29.0)
300	393	35.7 (29.1)	20.7 (28.2)	38.7 (28.3)	4.3 (8.0)	0.1 (0.2)	44.1 (21.6)	4.5 (11.6)	2.0 (6.3)	33.9 (23.9)	0.6 (1.4)	45.4 (24.4)	41.6 (26.0)
400	425	39.1 (28.4)	26.1 (29.5)	41.2 (27.4)	9.3 (10.2)	0.2 (0.3)	49.8 (17.7)	5.1 (13.3)	3.0 (8.7)	42.4 (22.0)	0.8 (2.0)	49.5 (21.4)	47.7 (22.7)
500	414	43.0 (26.7)	30.3 (29.8)	42.6 (26.7)	19.2 (14.1)	0.2 (0.5)	55.1 (12.1)	6.3 (14.8)	4.0 (11.2)	44.8 (20.4)	1.7 (6.4)	53.5 (18.7)	53.1 (17.3)
750	433	47.3 (24.4)	30.9 (29.9)	47.3 (24.2)	50.4 (21.6)	0.4 (1.0)	59.5 (2.6)	7.8 (17.2)	6.0 (14.3)	52.6 (14.3)	2.4 (7.1)	59.0 (23.3)	58.0 (9.6)
1000	441	49.5 (22.7)	33.9 (29.5)	50.8 (21.3)	52.4 (20.5)	0.7 (1.8)	60.0 (0.0)	8.1 (17.4)	6.8 (15.6)	56.4 (10.0)	3.4 (10.3)	90.4 (57.4)	59.2 (6.2)
Overall	2901	34.7 (29.4)	22.6 (28.8)	36.0 (28.9)	20.3 (25.3)	0.2 (0.9)	42.4 (24.4)	5.0 (13.4)	3.3 (10.4)	36.7 (25.1)	1.4 (5.6)	48.4 (39.0)	46.9 (23.9)

Table 8: Average time in seconds (standard deviation) for solving random instances when varying  $c$ 

$c$	tested inst.	Branch-and-bound			Branch-and-price			Pseudo-polynomial				Others	
		MTP	BISON	CVRPSEP	VANCE	BELOV	SCIP-BP	ONECUT	ARCFLOW	DPFLOW	VPSOLVER	BASIC ILP	CSTRPROG
50	223	27.2 (29.5)	9.4 (21.5)	23.2 (27.9)	20.2 (25.5)	0.0 (0.0)	43.0 (23.5)	0.0 (0.0)	0.0 (0.0)	13.6 (17.7)	0.0 (0.0)	48.7 (44.7)	42.6 (25.9)
75	240	26.1 (29.6)	14.8 (25.5)	26.4 (29.2)	20.2 (25.5)	0.0 (0.0)	42.0 (24.3)	0.0 (0.0)	0.0 (0.0)	19.5 (20.7)	0.0 (0.0)	53.2 (58.9)	41.1 (26.7)
100	234	32.0 (29.7)	14.9 (25.7)	31.1 (29.6)	20.3 (25.2)	0.0 (0.0)	42.8 (24.3)	0.0 (0.0)	0.1 (0.0)	26.4 (23.1)	0.0 (0.0)	50.5 (38.0)	45.0 (25.2)
120	241	33.5 (29.5)	17.8 (27.1)	33.7 (29.3)	19.7 (25.0)	0.0 (0.0)	42.2 (24.5)	0.1 (0.1)	0.1 (0.1)	29.6 (24.2)	0.1 (0.0)	49.8 (40.7)	45.1 (25.0)
125	251	30.0 (29.8)	18.0 (27.5)	30.0 (29.6)	19.2 (24.8)	0.0 (0.0)	41.0 (25.1)	0.1 (0.1)	0.1 (0.1)	30.9 (24.6)	0.1 (0.1)	50.4 (45.4)	44.3 (25.3)
150	240	35.1 (29.4)	19.3 (27.8)	38.2 (28.6)	19.5 (24.9)	0.0 (0.0)	41.8 (24.5)	0.1 (0.1)	0.2 (0.1)	34.9 (24.2)	0.1 (0.1)	50.9 (40.7)	47.3 (23.6)
200	246	37.6 (28.8)	22.8 (28.8)	39.2 (28.1)	20.5 (25.4)	0.0 (0.0)	40.8 (25.1)	0.3 (0.3)	0.3 (0.4)	39.3 (24.1)	0.2 (0.2)	51.4 (39.7)	48.7 (22.3)
300	237	39.0 (28.3)	26.8 (29.5)	41.4 (27.2)	21.9 (25.5)	0.1 (0.3)	44.4 (23.7)	1.5 (2.0)	1.3 (2.2)	45.6 (22.6)	0.6 (0.7)	48.1 (29.0)	46.5 (24.2)
400	245	36.7 (29.2)	30.5 (29.8)	40.4 (28.0)	20.3 (25.4)	0.2 (0.1)	41.6 (24.8)	3.9 (5.7)	2.8 (5.7)	47.4 (21.7)	1.1 (1.9)	47.0 (31.8)	48.8 (22.6)
500	243	38.3 (28.6)	29.5 (29.8)	41.8 (27.3)	20.7 (25.4)	0.3 (0.4)	44.8 (23.8)	8.6 (11.6)	5.1 (10.2)	49.2 (20.9)	1.9 (4.5)	45.7 (31.0)	50.0 (21.6)
750	249	40.8 (27.8)	32.0 (29.7)	43.4 (26.7)	21.0 (25.8)	0.8 (1.2)	42.4 (24.5)	18.6 (22.1)	11.2 (17.2)	49.9 (20.2)	4.3 (8.2)	43.1 (28.6)	52.0 (19.9)
1000	252	39.0 (28.4)	33.1 (29.8)	41.9 (27.4)	20.7 (25.6)	1.3 (2.3)	42.5 (24.8)	25.3 (25.1)	17.6 (21.9)	51.5 (18.9)	7.4 (14.8)	42.1 (27.5)	50.5 (21.3)
Overall	2901	34.7 (29.4)	22.6 (28.8)	36.0 (28.9)	20.3 (25.3)	0.2 (0.9)	42.4 (24.4)	5.0 (13.4)	3.3 (10.4)	36.7 (25.1)	1.4 (5.6)	48.4 (39.0)	46.9 (23.9)

Table 9: Average time in seconds (standard deviation) for solving random instances when varying weight range

Range	tested inst.	Branch-and-bound			Branch-and-price			Pseudo-polynomial				Others	
		MTP	BISON	CVRPSEP	VANCE	BELOV	SCIP-BP	ONECUT	ARCFLOW	DPFLOW	VPSOLVER	BASIC ILP	CSTRPROG
0.1 / 0.7	785	35.1 (29.1)	15.3 (25.9)	31.9 (29.3)	25.3 (25.2)	0.2 (0.4)	50.6 (19.9)	11.0 (19.7)	7.3 (16.2)	45.5 (22.2)	3.5 (9.8)	60.6 (35.1)	33.7 (28.1)
0.1 / 0.8	729	42.2 (27.2)	33.3 (29.4)	41.3 (27.2)	25.7 (26.5)	0.2 (0.6)	45.0 (23.5)	5.9 (13.7)	3.9 (10.3)	40.8 (23.9)	1.2 (3.8)	48.7 (28.5)	47.7 (23.2)
0.2 / 0.7	878	44.8 (25.8)	32.8 (29.6)	46.5 (24.7)	20.8 (25.6)	0.3 (1.4)	41.2 (25.3)	1.7 (5.5)	1.0 (2.9)	30.2 (25.3)	0.3 (0.6)	53.1 (47.5)	52.8 (19.0)
0.2 / 0.8	509	5.8 (17.4)	0.7 (6.2)	16.8 (26.7)	4.2 (13.8)	0.2 (0.4)	28.4 (24.3)	0.3 (0.7)	0.4 (1.3)	28.7 (25.0)	0.2 (0.3)	20.8 (25.8)	55.9 (14.9)
Overall	2901	34.7 (29.4)	22.6 (28.8)	36.0 (28.9)	20.3 (25.3)	0.2 (0.9)	42.4 (24.4)	5.0 (13.4)	3.3 (10.4)	36.7 (25.1)	1.4 (5.6)	48.4 (39.0)	46.9 (23.9)

the results are listed according to the number of items, in Table 5 according to the capacity, and in Table 6 according to the weight over capacity ratios. The entries give the same information as in Table 1. Similarly, Tables 7-9 report average CPU times and standard deviations with the same grouping policy. Globally, the results confirm observations 1.-4. made for the literature instances. We additionally observe that:

5. BELOV solved all instances within one minute, and it appears to be clearly superior to all other codes but VPSOLVER, which solved just 10 instances less (out of 2,901);
6. SCIP-BP is effective on small-size instances ( $n \leq 100$ );
7. the performance of branch-and-price algorithms is not affected by the capacity, while that of pseudo polynomial methods is. In particular, the behavior of ARCFLOW and DPFLOW depends on the three considered parameters, especially on the capacity and the item weights.

The best-in-class algorithms turned out to be the same as for the previous benchmark. In Table 10 (the counterpart of Table 3) we show the results obtained by the four selected computer codes within a time limit of 10 minutes, grouped by number of items. The results confirm the algorithms' ranking.

Table 10: Number of random instances solved in less than ten minutes when varying  $n$

n	tested inst.	BISON	BELOV	ARCFLOW	VPSOLVER
50	165	<b>165</b>	<b>165</b>	<b>165</b>	<b>165</b>
100	271	261	<b>271</b>	<b>271</b>	<b>271</b>
200	359	299	<b>359</b>	<b>359</b>	<b>359</b>
300	393	269	<b>393</b>	<b>393</b>	<b>393</b>
400	425	250	<b>425</b>	<b>425</b>	<b>425</b>
500	414	212	<b>414</b>	<b>414</b>	<b>414</b>
750	433	217	<b>433</b>	431	<b>433</b>
1000	441	200	<b>441</b>	434	<b>441</b>
Total	2901	1873	<b>2901</b>	2892	<b>2901</b>

As all the considered instances were solved to optimality, we used the four selected algorithms for a set of experiments on the new, difficult, ANI instances we have described in Section 6.1. In this case, each algorithm was given a time limit of one hour per instance. The outcome of the experiments is reported in Tables 11 (number of solved instances and average absolute gap with respect to the lower bound) and 12 (average CPU time and standard deviation). The results confirm that the ANI instances are indeed very difficult to solve to proven optimality. It turns out that the AI instances as well look quite hard, although a good heuristic, specially tailored wrt the special structure of these instances, is likely to find an optimal solution (whose optimality could then easily be proved), indicating that Non-IRUP plays an important role in this context.

Overall, our experiments show that, among the algorithms we tested, BELOV and VPSOLVER are the best ones. As both use quite complex tool, ARCFLOW can be seen as a reasonable compromise between simplicity and performance. Basic ILP and SCIP-BP can

Table 11: Number of difficult instances (ANI) solved in less than 1 hour (average gap wrt lower bound). The AI instances are included for the sake of comparison.

$n(\text{ANI})$	$n(\text{AI})$	$\bar{c}$	ARCFLOW		BISON		BELOV		VPSOLVER	
			ANI	AI	ANI	AI	ANI	AI	ANI	AI
201	202	2500	16 (0.7)	44 (0.1)	0 (1.0)	3 (0.9)	<b>50</b> (0.0)	<b>50</b> (0.0)	47 (0.1)	<b>50</b> (0.0)
402	403	10000	0 (1.0)	0 (1.0)	0 (1.0)	0 (1.0)	1 (1.0)	45 (0.1)	6 (0.9)	42 (0.2)
600	601	20000	-	-	-	-	0 (1.0)	21 (0.6)	0 (1.0)	8 (0.8)
801	802	40000	-	-	-	-	0 (1.0)	0 (1.0)	0 (1.0)	0 (1.0)
1002	1003	80000	-	-	-	-	-	-	-	-
Overall			16 (0.8)	44 (0.6)	0 (1.0)	3 (1.0)	51 (0.7)	116 (0.4)	53 (0.7)	100 (0.5)

Table 12: Average time in seconds (standard deviation) for solving difficult instances (ANI). The AI instances are included for the sake of comparison.

$n(\text{ANI})$	$n(\text{AI})$	$\bar{c}$	ARCFLOW		BISON		BELOV		VPSOLVER	
			ANI	AI	ANI	AI	ANI	AI	ANI	AI
201	202	2500	2723 (1376)	964 (1099)	3600 (0)	3384 (862)	144 (119)	91 (119)	415 (1056)	54 (128)
402	403	10000	3601 (0)	3601 (0)	3600 (0)	3600 (0)	3556 (321)	699 (1043)	3304 (846)	1130 (1201)
600	601	20000	-	-	-	-	3602 (3)	2539 (1321)	3600 (0)	3509 (293)
801	802	40000	-	-	-	-	3602 (5)	3601 (5)	3600 (0)	3600 (0)
1002	1003	80000	-	-	-	-	-	-	-	-
Overall			1581 (1064)	2943 (1269)	3600 (0)	3492 (616)	2726 (1504)	1733 (1639)	2730 (1504)	2073 (1653)

be used for small instances. Branch-and-bound algorithms MTP, CVRPSEP and, in particular, BISON can be an alternative when one wants to avoid the use of solvers. CSTRPROG is generally inefficient, but it has the advantage of easily allowing additional constraints. Although ONECUT is 35 years old, it is competitive with much more recent approaches. Among pseudo-polynomial approaches, DPFLOW has mainly theoretical interest, but has the advantage of being easily understandable. Among branch-and-price algorithms, VANCE has mainly historical interests, but it has an acceptable performance.

A final observation concerns the fact that the pseudo-polynomial approaches, that in the past were rarely directly used in practice as ILP formulations, turn out to be now extremely competitive. Table 13, produced thanks to the courtesy of IBM CPLEX, compares the performance of eight versions of the code (from CPLEX 6.0, dated 1998, to CPLEX 12.6.0,

Table 13: Number of selected instances solved [average time in seconds] using different versions of CPLEX.

Time	tested inst.	6.0 (1998)	7.0 (1999)	8.0 (2002)	9.0 (2003)	10.0 (2006)	11.0 (2007)	12.1 (2009)	12.6.0 (2013)
10 minutes	20	13 [366]	10 [420]	5 [570]	17 [268]	19 [162]	<b>20</b> [65]	19 [117]	<b>20</b> [114]
60 minutes	20	16 [897]	15 [1210]	15 [2009]	<b>20</b> [343]	<b>20</b> [186]	<b>20</b> [65]	19 [267]	<b>20</b> [114]

dated 2013) in the solution of the ILPs produced by ARCFLOW for 20 selected random instances. The instances had  $n$  ranging between 300 and 1000, and  $c$  ranging between 400 and 1000. The resulting ILPs had a number of rows (resp. columns) ranging between 482 and 1093 (resp. between 32 059 and 111 537). Each CPLEX version was run on a single core of an Intel Xeon Processor E5430 running at 2.66 GHz and equipped with 24 GB of memory, both with a time limit of 10 minutes and a time limit of one hour. The entries provide the number of instances solved to proven optimality and, in square brackets, the average CPU time.

The results in the first line show that, up to the early noughties, only a relatively small number of these instances could be solved within ten minutes, while the recent versions are very effective. The “irregular” behavior of the solver (previous versions give sometimes better results) is only apparently surprising. It is indeed known (see, e.g., Lodi [93] or Achterberg and Wunderling [1]) that, on specific instances, an older version of CPLEX can beat a newer one. In our case, the experiments were made on a small set of instances of a specific problem, so a fortiori irregularities could be expected. The second line shows that, in one CPU hour, about 75% of our instances could be solved prior to 2003 while the subsequent versions could solve practically all of them. The number of solved instances is in this case much more regular, but the average CPU time is not: compare, e.g., versions 11.0 and 12.6.0. Overall, by considering that ONECUT was developed in the mid-eighties, and ARCFLOW in the late nineties, these results well explain on one hand the choice of not pursuing their direct use, and on the other hand the good computational performance obtained nowadays.

## 7 Conclusions

We have reviewed the mathematical models and the exact algorithms developed in the last fifty years for one of the most famous combinatorial optimization problems. The bin packing problem and its main generalization (the cutting stock problem) have attracted many researchers, whose contributions have accompanied the development of algorithmic tools for the exact solution of combinatorial optimization problems. We have discussed the main approaches proposed in the literature, and we have provided an experimental evaluation of the available software on different classes of benchmarks, including a newly developed class of instances for which the exact algorithms can hardly obtain a provably optimal solution. We have additionally evaluated the influence that the improvement of ILP solvers has had on the performance of pseudo-polynomial formulations. The tested software and the benchmarks are now available in a dedicated library.

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